A Dynamic Model of Equilibrium with Private Information

Maxim Engers, Monica Hartmann, and Steven Stern

October 2020

1 Introduction

We present a finite-horizon, multiperiod model of a market for durable goods[1] (like cars) whose owners have private information about the good's quality. A good's quality changes stochastically over time in a way that need not be stationary. Equilibrium is characterized not only by prices and quantities, but also by the quality distributions of the different classes of the goods for sale. We use a fixed-point method to prove the existence of equilibrium.

2 Literature Review

The vast economic literature on markets with asymmetric information dates back to Akerlof (1970), which showed that, in a durable goods market, asymmetric information acts as an impediment to trade and even may have the potential to eliminate trade entirely. Akerlof's example was the used car market, but it was soon recognized that asymmetric information can play a critical role in many other contexts, including labor markets (Spence, 1973), insurance markets (Rothschild and Stiglitz, 1976) and credit markets (Stiglitz and Weiss, 1981). The role of asymmetric information in markets for financial assets has been a focus since the financial crisis of 2008. See, for example, Tirole (2012) and Guerrieri and Shimer (2014).

Akerlof formulated his original lemons model in a static framework. Hendel and Lizzeri (1999) extended the study of adverse selection in markets for durable goods to a dynamic model. Unlike our model, published work on dynamic analysis of durable goods with private information has typically used either a 2-period model or has focused on stationary equilibrium in models with infinitely many periods.

In some ways, the overall motivation for our work, to provide a tractable framework for empirical analysis, is most similar to that of Gillingham et al. (2019). A key difference is that sellers' private information is central to our

[1] Although, with minor modifications, the model can be applied to other durables (like housing), we use cars as our example below.

analysis, while their model assumes symmetrically informed buyers and sellers, following in the tradition of Rust (1985) and Berkovec (1985).

3 Model

There is a continuum H of consumers or households (we use the terms interchangeably), indexed by h. Each is one of a finite number of *types*, with consumers of the same type having the same observable characteristics. Consumers' types are indexed by $i \in I$, where I is a finite set.

Time is discrete, and the horizon is finite: time periods are indexed by $t \in \mathcal{T}$ where $\mathcal{T} = \{0, 1, 2, \ldots, T\}$. Consumers have a common discount factor $\beta \in (0, 1)$.[2] In each period, each consumer owns at most one car.

There is a continuum K of cars, indexed by k. Each is in one of a finite number of *classes*, with cars in the same class having the same publicly observable characteristics (such as year of manufacture and brand). Cars do not change their class as time elapses. Car classes are indexed by $j \in J$, where J is a finite set not including 0 or -1 that represents all car classes that are ever available at any time from 0 through T. We use the additional index $j = 0$ to denote the consumer's outside option of owning no car, and (with a slight abuse of terminology) we treat 0 as though it denotes a class of cars: Thus, saying that the consumer chooses a car of class 0 means that the consumer chooses to own no car. Let $\mathcal{J}_t$ denote the set of all classes of cars supplied in period t. Then $J_t = \mathcal{J}_t \cup \{0\}$ represents the set consisting of all car classes available for purchase in period t as well as the outside option. Later, we also use the index -1 to indicate the choice of retaining one's car.

There are four additive components that comprise the utility flow u_{hkt} that accrues in period t to a consumer h of type i from owning car k in class j.

a) v_{ijt} is deterministic and known to all consumers.

b) q_{kt} is the car quality[3] (or lemons) term, whose value is known only to the current owner of car k. The probability distribution of q_{kt} for all cars in class j for sale in period t is endogenous because it depends on the decisions of car-owners and is denoted by G_{jt}. Although these distributions are endogenous, our assumptions (about the initial quality distributions and about the conditional distributions that describe how quality changes stochastically from period to period) ensure that there is a compact interval Q that contains the support of all these distributions.

c) ε_{hjt} represents the idiosyncratic component of the value of the match between consumer h and a car of class j in period t. The current vector $(\varepsilon_{hjt})_{j \in J_t}$ is known to consumer h, and the ε_{hjt} are independently drawn from a continuous distribution F^{ε}_{ij} whose support is the real line. We assume that match values vary by individual consumer h (rather than being the same for all consumers of

[2] At the cost of more complicated notation, we could readily generalize to allow different types of consumers to have different discount factors.

[3] Although we use "quality" to refer to the q_{kt} term, it is really the *privately observed* component of quality. Other components of u_{hkt} (such as v_{ijt}) also reflect quality.

any given type i); otherwise demand would not be continuous. The match values vary by period t and are assumed to be independent across time periods in order to provide sufficient motivation to trade. The full-support assumption ensures a positive quantity demanded and a positive quantity supplied or scrapped at any price, and hence a positive volume of trade in every class of cars in every period that the class is available. We assume that the F^{ε}_{ij} distributions each have a finite mean, which, without further loss of generality, we normalize to zero by adjusting the v_{ijt} terms.

d) ξ_k is a car-specific term whose value is common knowledge to buyers and sellers but is not observed by the econometrician. Its presence explains why some cars are scrapped while others, indistinguishable by the econometrician, are sold at various positive prices. If, for each car k, the support of the distribution of ξ_k is finite, then including the ξ_k terms raises no additional theoretical issues regarding the question of whether equilibrium exists. Consumers regard cars with different values of ξ_k as being in different classes, but the econometrician observes only aggregates that lump together cars that are distinct from the consumers' point of view.[4] We make this finite-support assumption, and hence lose no further generality by omitting the ξ_k and writing

$$u_{hkt} = v_{ijt} + q_{kt} + \varepsilon_{hjt}.$$

For the outside option ($j = 0$), we assume that $v_{i0t} = 0$ for all i and t, and quality q_{kt} is always zero, so $u_{h0t} = \varepsilon_{h0t}$.

We allow for possible costs of disposing of a car, either by choosing to scrap it or sell it.[5] The cost of scrapping a car in class j is $\gamma_j \in \mathbb{R}$. We do not restrict γ_j to be positive, so as to allow the possibility that scrapping may yield parts which are worth more than the cost of disassembly.

The seller of a car in class j incurs a nonnegative transactions cost so that, if the price is p, the net amount the seller receives is $\nu_j(p) \leq p$.[6] We assume that, for each $j \in J$, ν_j is strictly increasing (a higher price nets more for the seller), continuous, and unbounded above (so $\nu_j(p) \to \infty$ as $p \to \infty$).

The terminal value $\bar{V}_{ijT+1}(q)$ gives the value to type i of ending the final period T owning a car in class j of quality q. We assume that the functions $\bar{V}_{ijT+1}$ are continuous and nondecreasing.

In each period t, a fraction $\delta_{jt} \in [0, 1]$ of the cars in each class j surviving from period $t - 1$ are exogenously scrapped and this probability of scrap-

[4] If the ξ_k terms can vary over time as well as with car class, as long as the entire sequence of ξ_{kt} is known from the start, we can use the same argument. If the sequence is not known in advance, thinking of distinct levels of ξ_{kt} as corresponding to different classes would require us to extend our model so that a car's class is no longer fixed, but can instead change from period to period. The simplest way to model the probabilities of switching from one class to another is to make them exogenously given constants unaffected by q_{kt}.

[5] The equilibrium existence result applies whether such costs are zero or strictly positive. With zero transactions costs, idiosyncratic taste shocks give consumers a strong incentive to sell their cars and buy different ones. Nevertheless, information asymmetries lead some consumers, in some periods, to keep their current vehicles.

[6] At the cost of further notational clutter, the scrapping and transactions costs could readily also be allowed to vary according to seller type.

ping is assumed to be the same for all cars in class j.[7] Car quality evolves stochastically from period to period. A surviving car of class j that was of quality $q_{kt-1} = q'$ in period $t-1$, now (in period t) has quality $q_{kt} = q$ distributed with conditional density $z_{jt}(q, q')$, and we assume that z_{jt} is a continuous function. The associated conditional distribution function of q given q' is $Z_{jt}(q, q\prime) = \int_{x \leq q} z_{jt}(x, q\prime) dx$. We assume that these conditional distributions each have compact support. We also assume that an increase in previous quality either leaves the distribution of current quality unchanged or else increases it in the sense of first-order stochastic dominance: Formally, for any q, if $q'' > q'$, then $Z_{jt}(q, q'') \leq Z_{jt}(q, q')$.

In addition to used cars supplied by their owners, we allow for external suppliers, who supply new cars and can also supply used cars imported from outside the model. Altogether, these external sources supply a quantity S^{ex}_{jt} of cars of class j. We assume that each S^{ex}_{jt} is a continuous function of the price vector $(p_{jt})_{j \in \mathcal{J}_t, t \in \mathcal{T}}$. Let F^{ex}_{jt} denote the given initial distribution of quality, q_{kt}, for these externally supplied cars.

Each owner decides whether to keep her car or to dispose of it (i.e., to sell or scrap it). Nonowners, including those who just decided to dispose of their car, each decide whether to buy, and, if so, from which class. Nonowners have beliefs[8] about the quality of cars for sale in each class j that, in equilibrium, coincide with the actual probability distributions G_{jt}. These distributions G_{jt} are endogenous because they depend on owners' decisions about disposing of their cars.

The market clears, trade (and, if chosen, scrapping) occurs, and sellers incur transactions costs. For each class $j \in J_t$ and each type of buyer i, the mass M_{ijt} of the set of consumers of type i who own cars in class j changes endogenously over time, as do the distributions F_{ijt} of q_{kt}, the quality terms of the cars these consumers own. The initial masses and distributions, i.e. M_{ij0} and F_{ij0}, are given.

We assume that the quality distributions of externally supplied cars F^{ex}_{jt} and the initial quality distributions F_{ij0} are continuous and have compact support. Because the $Z_{jt}(q, q')$ distributions have compact support and are monotonic in q', all the F_{ijt} have support that is a subset of some compact interval Q. A useful special case is Q finite (discrete quality with a finite number of levels), but the model also permits a continuum of qualities.

[7] We allow the possibility of exogenous scrapping to reflect the possibility that cars may be totalled (but also allow the possibility that no exogenous scrapping occurs by allowing all δ_{jt} to be 0. For simplicity, we model all exogenous scrapping in each period as occurring at the start of the period. Although the model can have cars entering and leaving, it keeps the set of consumers the same in all periods. We do so for simplicity only: nothing essential changes if we expand the model to allow for exogenous entry and exit by consumers.

[8] This formulation relies on the assumption that car class j is all that potential buyers observe (in particular, that buyers do not observe seller type i). If buyers do observe i, the model could be adapted by thinking of each pair (i, j) as a separate class. This would mean that a car's class can change endogenously from period to period, but we believe that our results extend in this case.

4 Equilibrium

As usual, equilibrium is characterized by individual optimization and market clearing. Market clearing can involve two possible complications: transactions costs and the option of scrapping. Selling a car at price p in class j in any period t yields a net gain of $\nu_j(p)$, while scrapping it yields $-\gamma_j$, so its owner is indifferent between scrapping and selling at the price $\underline{p}_j$ that satisfies $\nu_j(\underline{p}_j) = -\gamma_j$.[9] At any price below $\underline{p}_j$, owners would rather scrap than sell and so, because quantity demanded is positive, no price below $\underline{p}_j$ is possible in equilibrium.[10] At any price above $\underline{p}_j$ owners prefer selling to scrapping. Thus, in equilibrium, for all j and t such that $p_{jt} > \underline{p}_j$, the quantity of cars in class j supplied in period t is equal to the quantity demanded. If $p_{jt} = \underline{p}_j$, the excess supply of cars, if positive, is scrapped.

Individual optimization has each consumer/household in each period maximizing expected utility based on prices, the variables observed, and beliefs about the unobserved variables. In each period t, each owner decides whether to keep her car. If not (or if a consumer entered the period without a car), she decides whether to buy a car and, if so, in which class. Consumer h knows the current vector $(\varepsilon_{hjt})_{j\in J_t}$ and, if h owns car k, then h knows q_{kt}. But h does not know future vectors $(\varepsilon_{hjs})_{j\in J_s}$ (for $s > t$); nor does h know q_{kt} for cars k owned by others. The consumer relies on her beliefs about these unknowns, and, in equilibrium, these beliefs match the actual distributions. Because of the independence assumption, the distributions representing beliefs about future ε_{hjs} are exogenously given.

Beliefs about current quality terms q_{kt} are more complicated: the distribution of a prospective purchase's quality is conditional on the fact that the car is being offered for sale. In equilibrium, these beliefs match the quality distributions G_{jt} of cars actually offered for sale, which depend on endogenous consumer choices.

Before defining equilibrium, it is helpful to consider the analogous notion of competitive general equilibrium in a standard model (i.e., a model with no private information). An equilibrium then consists of prices and quantities (supplied and demanded). But, based on prices alone, one can check whether the associated quantities, together with the given prices, comprise an equilibrium. Thus, one can leave the quantities implicit and consider whether any given price vector is an equilibrium price vector. So, finding an equilibrium amounts to finding a suitable price vector. Thus, standard proofs of the existence of equi-

[9] To see that $\underline{p}_j$ is well defined, note first that, as $p \to -\infty$, $n_j(p) \leq p \to -\infty$. Also, as $p \to \infty$, $n_j(p) \to \infty$. Hence, for any $-\gamma_j$, there exist p_1 and p_2 such that $n_j(p_1) < -\gamma_j < n_j(p_2)$. Because the function n_j is continuous, the intermediate value theorem then guarantees that the equation $n_j(p) = -\gamma_j$ has a solution. Because n_j is strictly increasing, the solution is unique. If scrapping is costly, the price $\underline{p}_j$ can be negative.

[10] We assume that the exogenous suppliers have access to the same scrapping technology as car owners (or to technology that is at least as efficient), so they too would be unwilling to sell if the price were below $\underline{p}_j$.

librium leave the quantities implicit. They proceed by constructing a suitable mapping from a set of price vectors into itself and showing that this mapping has a fixed point. In our more complex setting that includes private information about quality, quantities can still be left implicit, but prices alone are no longer enough to determine an equilibrium because prices are now insufficient to determine individual choices (and hence demand). In addition to prices, we need the distribution of the privately known quality terms for each available purchasing option. To characterize an equilibrium, we now need to specify prices p_{jt} *and* quality distributions G_{jt} for each time period $t \in \mathcal{T}$ and each class j of cars available at t (i.e., each $j \in \mathcal{J}_t$). Thus an equilibrium is characterized by a vector $(p_{jt}, G_{jt})_{j \in \mathcal{J}_t, t \in \mathcal{T}}$ of price-distribution pairs, one pair for each class of cars in each period that the class is available.

We define a mapping Φ on a set X of such vectors and use the Schauder Fixed-Point Theorem, the infinite-dimensional extension of the Brouwer Fixed-Point Theorem, to show that a fixed point of the mapping exists. Our argument requires that the set on which Φ is defined be compact. We thus need to ensure that the set of possible prices considered and the set of possible quality distributions are compact.

Recall that, because of the possibility of scrapping, no equilibrium price p_{jt} below $\underline{p}_j$ is possible. There is no analogous, exogenously determined *upper* bound on equilibrium prices. So, to meet the compactness requirement, we choose an arbitrary $\bar{p}$ such that $\bar{p} > \underline{p}_j$ for all $j \in J$ and we restrict prices p_{jt} to the compact set $[\underline{p}_j, \bar{p}]$. We show below that we can choose $\bar{p}$ sufficiently large that any fixed point of the function is an equilibrium.

To ensure that the set of distributions is compact, we consider the set Σ of all probability distributions on the Borel sets in Q endowed with the topology of convergence in distribution. This topology can be characterized by what it means for a sequence $\{\mu_n\}$ of probability distributions in Σ to converge to a probability distribution μ in Σ. There are various equivalent possible characterizations, but a simple one is that μ_n converges to μ if and only if $\int_Q f d\mu_n \to \int_Q f d\mu$ for all continuous real-valued functions $f : Q \to \mathbb{R}$.[11] This last condition can be stated equivalently in terms of expectation: if E_n denotes expectation with respect to μ_n and E denotes expectation with respect to μ, the condition is that μ_n converges to μ if and only if $E_n f \to Ef$ for all continuous f. A standard result says that, with this topology, Σ is compact.[12]

[11] Because Q is a compact set, we can state the condition in terms of continuous functions rather than the more common and equivalent version in terms of bounded continuous functions. Another common equivalent way of stating the condition is in terms of the cumulative distribution function (cdf). If F_n is the cdf of μ_n and F is the cdf of μ, then, as $n \to \infty$, μ_n converges to μ if and only if for all $q \in Q$ at which F is continuous, $F_n(q) \to F(q)$.

[12] See, for example, chapter 15 in Aliprantis and Border, who show that, with the specified topology, Σ is not only compact but also metrizable and separable, so that continuity of functions on Σ or X can be characterized by sequences.

5 The Fixed-Point Mapping

The function Φ maps the set of all vectors of price-distribution pairs: $X = \prod_{t\in\mathcal{T}}\prod_{j\in\mathcal{J}_t}([\underline{p}_j,\bar{p}]\times\Sigma)$ into itself. X is the Cartesian product of nonempty, convex, and compact sets. It follows that X is itself nonempty, convex, and, by the Tychonoff Product Theorem, compact.

The function Φ is generated by a procedure that we provide below. The procedure has two stages: a backward stage and then a forward stage. First, working backwards from the final period T to the initial period 0, it recursively determines various value functions and choice-probability functions. Then, starting at period 0 and working forwards to period T, it determines demands and generates the updated quality distributions G_{jt}^{new} and updated prices p_{jt}^{new}. Because the procedure consists of a sequence of instructions, we hope the reader will forgive our use of the imperative mood in specifying it.

In order to include the option of owning no car (case $j = 0$), we adopt the following conventions that set the depreciation, price, cost, and quality of the zero option to zero: for all t, $\delta_{0t} = 0$, $p_{0t} = 0$, $n_0(0) = 0$, and G_{0t} and $Z_{0t}(q, 0)$ are set to the degenerate distribution that puts all probability mass at zero.

Recall that the terminal value functions, $\bar{V}_{ijT+1}(q)$ denoting the value to type i of ending the final period T owning a car in class j of quality q, are given. For $t = T, T-1, \ldots 2, 1$, we recursively define the various terms needed for calculating the updated quality distributions and prices.

5.1 The Backward Stage

Based on the input vector $(p_{jt}, G_{jt})_{j\in\mathcal{J}_t, t\in\mathcal{T}}$, start in the final period, and work backwards, period by period. In each period t, for each consumer type i and each class j of car, determine the value function $V_{ijt}(q)$ that represents the expected value (excluding the match term ε_{hjt}) to type i of owning and keeping a car of class j as a function of its quality $q \in Q$:

$$V_{ijt}(q) = v_{ijt} + q + \beta[\delta_{jt}\bar{V}_{i0t+1}(0) + (1-\delta_{jt})\int_Q \bar{V}_{ijt+1}(q_{t+1}) z_{jt+1}(q_{t+1}, q) dq. \quad (1)$$

Next determine W_{ijt}, the expected value (again excluding ε_{hjt}) to type i of acquiring a car in class j. This is just the expectation of $V_{ijt}(q)$ taken with respect to the quality distribution G_{jt} of cars in class j for sale in period t:

$$W_{ijt} = \int_Q V_{ijt}(q) dG_{jt}(q). \quad (2)$$

Then determine the function $\bar{V}_{ijt}(q)$ that, for each quality q, gives the expected value to type i of entering period t with a car in class j. This reflects the best option between keeping the car and disposing of it and choosing among any of the options $j' \in J_t$. The utility from keeping the car includes the match term

ε_{hjt}, while the utility from switching to j' includes $\varepsilon_{hj't}$, and so the expectation is taken with respect to the distribution of the ε terms. Thus,

$$\bar{V}_{ijt}(q) = E\max\{V_{ijt}(q) + \varepsilon_{hjt}, \max_{j' \in J_t}\{W_{ij't} + \varepsilon_{hj't} + \nu_j(p_{jt}) - p_{j't}\}\}, \quad (3)$$

where the operator E denotes expectation with respect to the distribution of the vector $(\varepsilon_{hj't})_{j' \in J_t}$.

The final step in the backward part of the procedure is to determine various choice-probability functions. A consumer h of type i who owns car k in class j of quality q has the following $|J_t| + 1$ options: Either keep the car, yielding $V_{ijt}(q) + \varepsilon_{hjt}$ or dispose of the car and choose class $j' \in J_t$, yielding $W_{ij't} + \varepsilon_{hj't} + \nu_j(p_{jt}) - p_{j't}$. The probability that each of the $|J_t| + 1$ choices is optimal is given by the likelihood that the associated term is the maximum.[13] Because the match terms $\varepsilon_{hj't}$ for $j' \in J_t$ are continuously distributed and not perfectly correlated, ties occur with probability zero, and so the optimal choice is almost always unique, with only the one possible following exception: there can be a positive probability of a tie between keeping a car of class j and buying another from the *same* class (because the utility from each of these choices involves the same match term ε_{hjt}). Because V_{ijt} is strictly increasing, this positive probability of a tie occurs (if at all) for a single value of $q \in Q$. This means that all the choice probabilities are uniquely defined except for at most a single $q \in Q$. Thus the procedure uniquely determines the equivalence class that includes functions that differ only on a set of measure zero, which is all we require for the procedure that determines the fixed-point mapping.

Accordingly, let $\mathbb{P}^{j'}_{ijt}(q)$ denote the probability that a household of type i sells its car of class j and quality q and buys one of class j' in period t.[14] Thus, $\mathbb{P}^{j'}_{ijt}(q)$ is the probability of the event:

$$W_{ij't}+\varepsilon_{hj't}+\nu_j(p_{jt})-p_{j't} \geq \max\{V_{ijt}(q)+\varepsilon_{hjt}, \max_{j'' \in J_t}\{W_{ij''t}+\varepsilon_{hj''t}+\nu_j(p_{jt})-p_{j''t}\}\}. \quad (4)$$

Finally, the probability that a household of type i keeps its car of class j and quality q in period t is:

$$\mathbb{P}^{-1}_{ijt}(q) = 1 - \sum_{j' \in J_t} \mathbb{P}^{j'}_{ijt}(q). \quad (5)$$

[13] If j and j' are both 0, the option of keeping the car is indistinguishable from the option of selling the car of type 0 and then buying another car of type 0. Thus, for non-owners (with $j = 0$), there are really only $|J_t|$ distinct options, and we do not include the -1 option in this case. In other words, the recursion formulae below treat consumers who remain without a car from period to period as if they were selling their type-zero car and buying a different one. Since all type-zero purchases involve zero transactions costs and identical utility consequences, no results in the model depend on this choice.

[14] Recall that we allow j and j' to take the value 0. When $j'$is zero, $\mathbb{P}^{0}_{ijt}(q)$ is the probability that no car is purchased. When j is zero, $\mathbb{P}^{j'}_{i0t}(q)$ is the probability that a household that does not own a car buys a car of class j'.

This completes the backward part of the procedure.

5.2 The Forward Stage

Now, we describe the forward part, showing how to obtain the updated distributions of quality of cars for sale, G_{jt}^{new}, and to determine demands, which are then used to generate the updated prices, p_{jt}^{new} . To do so, we start at period $t = 0$ and use the choice probabilities obtained in the backward procedure to determine the masses M_{ijt} and distributions F_{ijt}. Recall that these masses and distributions are given for period $t = 0$, and we compute them successively for $t = 1, 2, 3 \ldots, T$.

At the start of each period $t > 0$, a fraction δ_{jt} of the cars of class j from the previous period are exogenously scrapped. Let $M_{ijt}^{/}$ denote the mass of consumers of type i in period t who own a car in class j after this scrapping. The owners whose cars were exogenously scrapped now own no car, so

$$M_{i0t}^{/} = M_{i0t-1} + \sum_{j \in \mathcal{J}_{t-1}} \delta_{jt} M_{ijt-1}. \tag{6}$$

For $j \in \mathcal{J}_{t-1}$,

$$M_{ijt}^{/} = (1 - \delta_{jt}) M_{ijt-1}. \tag{7}$$

Surviving cars that were of quality q_{t-1} have new quality q distributed with conditional density $z_{jt}(q, q_{t-1})$. Thus, before consumers make their sales decisions in period t, the density of the quality terms of cars in class j owned by consumers of type i is

$$f_{ijt}^{b}(q) = \int_Q z_{jt}(q, q_{t-1}) dF_{ijt-1}(q_{t-1}). \tag{8}$$

The mass of consumers of type i who buy a car of class j in period t is

$$B_{ijt} = \sum_{j' \in J_{t-1}} M_{ij't}^{/} \int_Q \mathbb{P}_{ij't}^{j}(q) f_{ij't}^{b}(q) dq, \tag{9}$$

while the mass of consumers of type i who retain their car of class j from period $t-1$ through period t is

$$R_{ijt} = M_{ijt}^{/} \int_Q \mathbb{P}_{ijt}^{-1}(q) f_{ijt}^{b}(q) dq. \tag{10}$$

The mass of consumers of type i who sell their car of class j in period t is $M_{ijt}^{/} - R_{ijt}$ so that the total quantity of class j cars supplied in period t is

$$S_{jt} = \sum_{i \in I} (M_{ijt}^{/} - R_{ijt}) + S_{jt}^{ex}. \tag{11}$$

The total quantity of class j cars demanded in period t is

$$D_{jt} = \sum_{i \in I} B_{ijt}. \tag{12}$$

The updated quality distributions of cars of class j in period t (an output of the function Φ) is

$$G_{jt}^{new}(q) = \frac{\sum_{i \in I} M_{ijt}^{/} \int_{x \leq q} (1 - \mathbb{P}_{ijt}^{-1}(x)) f_{ijt}^{b}(x) dx + S_{jt}^{ex} F_{jt}^{ex}(q)}{S_{jt}}. \tag{13}$$

The mass of consumers of type i who choose to own a car in class j in period t is

$$M_{ijt} = B_{ijt} + R_{ijt}, \tag{14}$$

and the corresponding distribution describing the quality of the cars these consumers own, is given by

$$F_{ijt}(q) = \frac{B_{ijt} G_{jt}^{new}(q) + M_{ijt}^{/} \int_{x \leq q} \mathbb{P}_{ijt}^{-1}(x) f_{ijt}^{b}(x) dx}{M_{ijt}}. \tag{15}$$

Finally, the updated price of a car of class j in period t (the other output of the function Φ) is

$$p_{jt}^{new} = \max\{\underline{p}_j, \min\{\bar{p},\ p_{jt} + D_{jt} - S_{jt}\}\}. \tag{16}$$

This price-adjustment mechanism increases the prices of goods in excess demand and decreases the prices of all goods in excess supply, subject to the requirement that prices remain within the fixed bounds $\underline{p}_j$ and $\bar{p}$. If excess demand is zero the price is unchanged.

By finite induction, we establish that, if a sequence of initial values G_{jt}^{n} weakly converges to limit G_{jt}^{0} and a sequence of price vectors p_{jt}^{n} converges to price vector p_{jt}^{0}, then, as $n \to \infty$, at each stage of our procedure, the limit of the sequence of functions or values calculated starting with $x^n = (p_{jt}^{n}, G_{jt}^{n})_{j \in \mathcal{J}_t, t \in \mathcal{T}}$ converges to the functions or values calculated starting with $x^0 = (p_{jt}^{0}, G_{jt}^{0})_{j \in \mathcal{J}_t, t \in \mathcal{T}}$.

In Section 7, by verification of the sequential continuity of each step of the backward and forward part of the construction of the function Φ, we establish that Φ is a continuous function from X to itself. Thus, Φ has a fixed point, and, for $\bar{p}$ large enough, any fixed point is an equilibrium.

6 Some Properties of the Value Functions

A few properties of the value functions follow easily by finite induction: Because $Z_{jt+1}(q_{t+1}, q)$ is nonincreasing in q and $\bar{V}_{ijT+1}$is nondecreasing, V_{ijt} is strictly increasing and hence $\bar{V}_{ijt}$ is nondecreasing. Also V_{ijt} and $\bar{V}_{ijt}$ are both continuous (in q). To see that V_{ijt} is continuous, we need to show that

$$\int_Q \bar{V}_{ijt+1}(q_{t+1})z_{jt+1}(q_{t+1}, q)dq_{t+1}$$

is continuous, and it is sufficient to show that, for any sequence $q^n \to q^0$, we have

$$\int_Q \bar{V}_{ijt+1}(q_{t+1})z_{jt+1}(q_{t+1}, q^n)dq_{t+1} \to \int_Q \bar{V}_{ijt+1}(q_{t+1})z_{jt+1}(q_{t+1}, q^0)dq_{t+1}.$$

Because z_{jt+1} is bounded above, and Q, being compact, has finite Lebesgue measure, the Dominated Convergence Theorem applies, and continuity of V_{ijt} follows. To see that $\bar{V}_{ijt}$ is continuous, we note that $V_{ijt}(q)$ and $\nu_j(p_{jt})$ are bounded and so there is a constant C such that, for any vector $(\varepsilon_{hj't})_{j' \in J_t}$,

$$\max\{V_{ijt}(q)+\varepsilon_{hjt}, \max_{j' \in J_t}\{W_{ij't}+\varepsilon_{hj't}+\nu_j(p_{jt})-p_{j't}\}\} \leq C+|\varepsilon_{hjt}|+\sum_{j' \in J_t} |\varepsilon_{hj't}|.$$

The right hand side has finite expectation because each ε term has finite expectation. (Recall that we assume that ε has finite expectation, and so its absolute value $|\varepsilon|$ has finite expectation too.) The left hand side is continuous in q, and thus, taking expectation, the continuity of $\bar{V}_{ijt}$ follows from the Dominated Convergence Theorem.

7 Showing that the Fixed-Point Mapping is Continuous

It is helpful to think of the procedure as proceeding via a sequence of stages: a new stage occurs each time, in each period, one of the numbered equations is referenced. Each of the numbered equations can be viewed as a mapping which maps the various real variables and functions (of q) on the right-hand side to produce as output the function (again of q) or real variable on the left-hand sided. For example, recall equation (1):

$$V_{ijt}(q) = v_{ijt} + q + \beta[\delta_{jt}\bar{V}_{i0t+1}(0) + (1-\delta_{jt})\int_Q \bar{V}_{ijt+1}(q_{t+1})z_{jt+1}(q_{t+1}, q)dq.$$

Here, the left hand-side is a function, $V_{ijt}(q)$, and the inputs to the mapping are the real variable $\bar{V}_{i0t+1}(0)$ and the function $\bar{V}_{ijt+1}(q_{t+1})$. In equation (2),

$$W_{ijt} = \int_Q V_{ijt}(q)dG_{jt}(q),$$

the left hand-side is a real variable, and the inputs to the mapping are the functions $V_{ijt}(q)$ and $G_{jt}(q)$. Thus each line can be thought of as a mapping $\Gamma_m : A_m \to B_m$.

We show that the mapping at each stage is *sequentially continuous.* A mapping $\Gamma : A \to B$ is sequentially continuous if, for every sequence $\{a_n\}$ in A, $a_n \to a_0 \Rightarrow \Gamma(a_n) \to \Gamma(a_0)$, i.e. if the sequence $\{a_n\}$ converges to a_0, then $\Gamma(a_n)$ converges to $\Gamma(a_0)$. Continuity of a mapping always implies sequential continuity and, if A is a metric space (or, more generally a first-countable space), the converse also holds so that continuity and sequential continuity are equivalent. It follows almost immediately from the definition that the composition of two sequentially continuous mappings is sequentially continuous: If $\Gamma_1 : A_1 \to B_1$ and $\Gamma_2 : A_2 \to B_2$ are sequentially continuous and $B_1 \subset A_2$, then the composite mapping, $\Gamma_2 \circ \Gamma_1$, is sequentially continuous. Thus, by induction, the composition of any number of sequentially continuous mappings is sequentially continuous.

For a sequence of scalars (for example, prices p_{jt} or masses M_{ijt}) the meaning of convergence is unambiguous, but spaces of functions (for example, values $V_{ijt}(q)$ or choice probabilities $\mathbb{P}^{j'}_{ijt}(q)$, both defined for all $q \in Q$), admit many notions of convergence. For each mapping specified by the right-hand side of a numbered equation, we must choose a criterion that is weak enough to allow proof of the required convergence, yet strong enough to ensure the convergence of the scalars and functions constructed subsequently in the procedure. For the value functions $V_{ijt}(q)$ and $\bar{V}_{ijt}(q)$, we use uniform convergence on the set Q. (Using uniform convergence here amounts to using the supremum norm $\sup_{q\in Q}|V(q) - \tilde{V}(q)|$ to determine the distance between any two functions V and $\tilde{V}$.) For the choice-probability functions $\mathbb{P}^{j'}_{ijt}(q)$ and $\mathbb{P}^{-1}_{ijt}(q)$, we consider equivalence classes of all such functions that are equal almost everywhere on Q with regard to standard Lebesgue measure, and the notion of convergence is the essential supremum norm or $||_\infty$, so that the distance between any two (equivalence classes of) functions $\mathbb{P}$ and $\tilde{\mathbb{P}}$ is $\inf\{\lambda > 0 : |\mathbb{P}(q) - \tilde{\mathbb{P}}(q)| \le \lambda$ almost everywhere$\}$. For the density functions, $f^b_{ijt}(q)$, and the distribution functions, $F_{ijt}(q)$, we use pointwise convergence and, for $G^{new}_{jt}(q)$ and $G_{jt}(q)$, we use convergence in distribution. With these convergence criteria, we verify that each of the numbered equations used in determining the fixed-point mapping defines a sequentially continuous mapping.

First consider equation (1) above. We can assume that $\bar{V}^n_{i0t+1}(0)$ converges to $\bar{V}^0_{i0t+1}(0)$, and that $\bar{V}^n_{ijt+1}(q_{t+1})$ converges uniformly to $\bar{V}^0_{ijt+1}(q_{t+1})$, or, equivalently that $\sup_{q_{t+1}\in Q}|\bar{V}^n_{ijt+1}(q_{t+1})-\bar{V}^0_{ijt+1}(q_{t+1})|$ converges to zero, and so $\sup_{q\in Q}|V^n_{ijt}(q)-V^0_{ijt}(q)|$ is bounded above by

$$\beta|\delta_{jt}(\bar{V}^n_{i0t+1}(0)-\bar{V}^0_{i0t+1}(0))|+(1-\delta_{jt})\int_Q \sup|\bar{V}^n_{ijt+1}(q_{t+1})-\bar{V}^0_{ijt+1}(q_{t+1})|z_{jt+1}(q_{t+1},q)dq|,$$

which converges to zero as required.

Next consider equation (2): $W_{ijt} = \int_Q V_{ijt}(q) dG_{jt}(q)$. Assuming $V_{ijt}^n(q)$ converges uniformly to $V_{ijt}^0(q)$ and $G_{jt}^n(q)$ converges in distribution to $G_{jt}^0(q)$, we need to show that

$$\int_Q V_{ijt}^n(q) dG_{jt}^n(q) - \int_Q V_{ijt}^0(q) dG_{jt}^0(q) \to 0.$$

The left-hand side can be written as

$$\int_Q [V_{ijt}^n(q) - V_{ijt}^0(q)] dG_{jt}^n(q) + \int_Q V_{ijt}^0(q) dG_{jt}^n(q) - \int_Q V_{ijt}^0(q) dG_{jt}^0(q). \qquad (17)$$

Because $V_{ijt}^0(q)$ is continuous and $G_{jt}^n \to G_{jt}^0$ in distribution,

$$\int_Q V_{ijt}^0(q) dG_{jt}^n(q) \to \int_Q V_{ijt}^0(q) dG_{jt}^0(q),$$

or, equivalently

$$\int_Q V_{ijt}^0(q) dG_{jt}^n(q) - \int_Q V_{ijt}^0(q) dG_{jt}^0(q) \to 0.$$

Because $V_{ijt}^n(q) - V_{ijt}^0(q)$ converges uniformly to zero, the first integral in (17) also converges to 0, and the result follows.

For (3) $\bar{V}_{ijt}(q) = E \max\{V_{ijt}(q) + \varepsilon_{hjt}, \max_{j' \in J_t}\{W_{ij't} + \varepsilon_{hj't} + \nu_j(p_{jt}) - p_{j't}\}\}$, we must show that, if $V_{ijt}^n(q)$ converges uniformly to $V_{ijt}^0(q)$, $W_{ij't}^n \to W_{ij't}^0$ and $p_{jt}^n \to p_{jt}^0$ for $j' \in J_t$, then

$$E \max\{V_{ijt}^n(q) + \varepsilon_{hjt}, \max_{j' \in J_t}\{W_{ij't}^n + \varepsilon_{hj't} + \nu_j(p_{jt}^n) - p_{j't}^n\}\}$$

converges uniformly to

$$E \max\{V_{ijt}^0(q) + \varepsilon_{hjt}, \max_{j' \in J_t}\{W_{ij't}^0 + \varepsilon_{hj't} + \nu_j(p_{jt}^0) - p_{j't}^0\}\}.$$

For any vector $(\varepsilon_{hj't})_{j' \in J_t}$,

$$\max\{V_{ijt}^n(q) + \varepsilon_{hjt}, \max_{j' \in J_t}\{W_{ij't}^n + \varepsilon_{hj't} + \nu_j(p_{jt}^n) - p_{j't}^n\}\}$$

converges uniformly to

$$\max\{V_{ijt}^0(q) + \varepsilon_{hjt}, \max_{j' \in J_t}\{W_{ij't}^0 + \varepsilon_{hj't} + \nu_j(p_{jt}^0) - p_{j't}^0\}\}.$$

Taking expectations gives the required result.

In (4) $\mathbb{P}_{ijt}^{j'}(q)$ is the probability of the event,

$$W_{ij't} + \varepsilon_{hj't} + \nu_j(p_{jt}) - p_{j't} \geq \max\{V_{ijt}(q) + \varepsilon_{hjt}, \max_{j'' \in J_t}\{W_{ij''t} + \varepsilon_{hj''t} + \nu_j(p_{jt}) - p_{j''t}\}\}.$$

Recall that ties can occur with positive probability for at most one value of q. So these probabilities and the probability $\mathbb{P}_{ijt}^{-1}(q)$ of retaining the car are uniquely determined almost everywhere. Fix i, j, t, and j'. To denote the nth term in the sequence of values $\mathbb{P}_{ijt}^{j'}(q)$, we use the notation $\mathbb{P}_{ijt}^{j'(n)}(q)$. So, $\mathbb{P}_{ijt}^{j'(n)}(q)$ is the probability of the event,

$$W_{ij't}^{n}+\varepsilon_{hj't}+\nu_j(p_{jt}^{n})-p_{j't}^{n} \geq \max\{V_{ijt}^{n}(q)+\varepsilon_{hjt}, \max_{j''\in J_t}\{W_{ij''t}^{n}+\varepsilon_{hj''t}+\nu_j(p_{jt}^{n})-p_{j''t}^{n}\}\},$$

and $\mathbb{P}_{ijt}^{j'(0)}(q)$ is the probability of the analogous event,

$$W_{ij't}^{0}+\varepsilon_{hj't}+\nu_j(p_{jt}^{0})-p_{j't}^{0} \geq \max\{V_{ijt}^{0}(q)+\varepsilon_{hjt}, \max_{j''\in J_t}\{W_{ij''t}^{0}+\varepsilon_{hj''t}+\nu_j(p_{jt}^{0})-p_{j''t}^{0}\}\},$$

in which all the n superscripts are replaced by 0 superscripts. Suppose $V_{ijt}^{n}(q)$ converges uniformly to $V_{ijt}^{0}(q)$, $W_{ij't}^{n} \to W_{ij't}^{0}$,and $p_{j't}^{n} \to p_{jt}^{0}$ for $j' \in J_t$. We show that, except on a set of measure zero, $\mathbb{P}_{ijt}^{j'(n)}(q)-$ $\mathbb{P}_{ijt}^{j'(0)}(q)$ converges uniformly to 0.

Using the continuity of the ν_j functions, for any $\epsilon > 0$ we can choose N such that $n > N$ ensures that

$|W_{ij't}^{n} - W_{ij't}^{0}| < \epsilon/2$, $|\nu_j(p_{jt}^{n}) - p_{j't}^{n} - (\nu_j(p_{jt}^{0}) - p_{j't}^{0})| < \epsilon/2$ and, for all $q \in Q$, $|V_{ijt}^{n}(q) - V_{ijt}^{0}(q)| < \epsilon$.

Thus, as long as $n > N$, the only way that there can be a difference in the optimal choice between the case with the superscript n values and the limit case with the superscript 0 values is if the corresponding $\varepsilon_{hj't}$ terms differ by an amount that is smaller than ϵ in absolute value. But since the $\varepsilon_{hj't}$ terms are continuously distributed, the probability of there being such a difference converges uniformly to zero as ϵ converges to zero.

For (6) and (7), because $M_{i0t}^{/} = M_{i0t-1}+\sum_{j\in\mathcal{J}_{t-1}} \delta_{jt}M_{ijt-1}$ and $M_{ijt}^{/} = (1-\delta_{jt})M_{ijt-1}$ are just linear combinations of previously determined real variables, the required convergence follows immediately.

Recalling (6) $f_{ijt}^{b}(q) = \int_Q z_{jt}(q, q_{t-1})dF_{ijt-1}(q_{t-1})$, since we can assume $F_{ijt-1}^{n}(q_{t-1})$ converges pointwise to $F_{ijt-1}^{0}(q_{t-1})$, it converges in distribution *a fortiori*. Because z_{jt} is continuous, for each $q \in Q$, the integral $\int_Q z_{jt}(q, q_{t-1})dF_{ijt-1}^{n}(q_{t-1})$ converges to $\int_Q z_{jt}(q, q_{t-1})dF_{ijt-1}^{o}(q_{t-1})$, which is the pointwise convergence required. Note that z_{jt}, being continuous on compact set $Q \times Q$, is bounded above by B, say, and so

$$f_{ijt}^{b}(q) = \int_Q z_{jt}(q, q_{t-1})dF_{ijt-1}(q_{t-1}) \leq \int_Q BdF_{ijt-1}(q_{t-1}) = B$$

In (9) $B_{ijt} = \sum_{j'\in J_{t-1}} M_{ij't}^{/}\int_Q \mathbb{P}_{ij't}^{j}(q)f_{ij't}^{b}(q)dq$, the convergence of the $\mathbb{P}_{ij't}^{j(n)}(q)$ in the essential sup norm ensures that the integrand converges pointwise (except on a set of measure zero). Since the $\mathbb{P}_{ij't}^{j(n)}(q)$ are bounded above by

1 and (as just shown) $f^b_{ij't}(q)$ is also bounded above, the required convergence follows by the Dominated Convergence Theorem.

For (10) $R_{ijt} = M^{/}_{ijt}\int_Q \mathbb{P}^{-1}_{ijt}(q) f^b_{ijt}(q)dq$. Exactly the same argument applies as in (9) above.

(11) and (12) $S_{jt} = \sum_{i\in I}(M^{/}_{ijt} - R_{ijt}) + S^E_{jt}$ and $D_{jt} = \sum_{i\in I} B_{ijt}$. Again, as in (6) above, the required convergence follows immediately.

(13) $G^{new}_{jt}(q) = \frac{\sum_{i\in I} M^{/}_{ijt}\int_{x\le q}(1-\mathbb{P}^{-1}_{ijt}(x))f^b_{ijt}(x)dx + S^E_{jt}F^E_{jt}(q)}{S_{jt}}$ For each q, the convergence of the sum $\sum_{i\in I} M^{/}_{ijt}\int_{x\le q}(1-\mathbb{P}^{-1}_{ijt}(x))f^b_{ijt}(x)dx$ follows by the same argument as in (7) above. Adding $S^E_{jt}F^E_{jt}(q)$ and dividing by S_{jt} preserves the convergence property because addition and division by a nonzero scalar are continuous operations. Since we have convergence for each q, pointwise convergence is established, implying convergence in distribution.

(14) $M_{ijt} = B_{ijt} + R_{ijt}$ converges as in (6).

(15) The pointwise convergence of

$$F_{ijt}(q) = \frac{B_{ijt}G^{new}_{jt}(q) + M^{/}_{ijt}\int_{x\le q}\mathbb{P}^{-1}_{ijt}(x)f^b_{ijt}(x)dx}{M_{ijt}}$$

follows by the same arguments used in (13) above.

Finally, since (16) $p^{new}_{jt} = \max\{\underline{p}_j, \min\{\bar{p}, p_{jt} + D_{jt} - S_{jt}\}\}$ gives new prices as a continuous function of real variables the sequential continuioty follows immediately.

8 Existence of Equilibrium

The previous section establishes that, at each stage of the procedure, the function from X to the space of functions or scalars constructed at that stage is sequentially continuous. In particular, the map from X that generates G^{new}_{jt} is sequentially continuous. Because X is metrizable, the map is continuous. The function Φ satisfies all the requirements of the Schauder Fixed-Point Theorem. This theorem says that, if X is a nonempty, compact, convex subset of a locally convex Hausdorff space, and if the function $\Gamma : X \to X$ is continuous, then the set of fixed points of Γ is nonempty and compact. Hence a fixed point exists.

We finally show that if $\bar{p}$ is sufficiently large then any fixed point is an equilibrium. Because of the way the function assigns prices, it is sufficient to show that if, at a fixed point, any price p_{jt} is at the maximum level $\bar{p}$ then there is not an excess demand for cars in class j in period t. Ruling out excess demand at arbitrarily high prices is not entirely straightforward here: Quantity demanded is strictly positive, no matter how high the price, because the support of the distribution of the match terms ε_{hjt} is not bounded above. This would not matter, if quantity supplied were bounded away from zero, but, because of the possibility of endogenous scrapping and of endogenous supply, there is no natural, strictly positive, lower bound on quantity. Nevertheless, there are only two possible ways that quantity can be close to zero in period t: In some prior period $t' < t$, there either was endogenous scrapping, or the quantity supplied was close to zero. In either case the price in period t' would have to be sufficiently *low*. Since $p_{jt} = \bar{p}$, anyone buying at the low price $p_{jt'}$ in period t' and selling at price $\bar{p}$ in period t can achieve an expected lifetime utility level of order $\bar{p}$. But, by the second lemma in the appendix, at any price vector, the average utility is bounded above by a constant independent of $\bar{p}$. Thus the fraction of consumers receiving a utility level of order $\bar{p}$ is (at most) of order $1/\bar{p}$. By choosing $\bar{p}$ sufficiently large we can thus infer that there is no excess demand for any good at the fixed point, and hence an equilibrium exists.

9 Conclusion

The existence result is worthwhile in its own right. For us, the most appealing aspect of the proof is that it outlines a method that promises to allow the actual computation of the equilibrium in models with persistent private information of the kind described.

References

[1] Akerlof, George A., "The Market for 'Lemons': Quality Uncertainty and the Market Mechanism." *Quarterly Journal of Economics*, 84(3), pp. 488-500, 1970.

[2] Aliprantis, Charalambos D., and Kim C. Border, Infinite Dimensional Analysis: A Hitchhiker's Guide, 3rd ed. New York: Springer, 2006.

[3] Berkovec, James, "New Car Sales and Used Car Stocks: A Model of the Automobile Market." *RAND Journal of Economics*, Vol. 16, No. 2. (Summer), pp. 195-214, 1985.

[4] Debreu, Gerard, Theory of Value, Yale University Press, New Haven, CT, 1959.

[5] Gillingham, Kenneth, Fedor Iskhakov, Anders Munk-Nielsen, John P. Rust and Bertel Schjerning, "Equilibrium Trade in Automobile Markets" (May 2019). NBER Working Paper No. w25840. Available at SSRN: https://ssrn.com/abstract=3390988

[6] Guerrieri, Veronica and Robert J. Shimer, "Dynamic Adverse Selection: A Theory of Illiquidity, Fire Sales, and Flight to Quality" *American Economic Review*, vol. 104 (7): 1875-1908, 2014

[7] Hendel, Igal and Alessandro Lizzeri, "Adverse Selection in Durable Goods Markets," *American Economic Review*, vol. 89 (5): 1097-1115, 1999

[8] Rothschild, Michael and Joseph Stiglitz, "Equilibrium in Competitive Insurance Markets: An Essay on the Economics of Imperfect Information," *Quarterly Journal of Economics*, vol. 90(4), pages 629-649, 1976.

[9] Rust, John, "Stationary equilibrium in a market for durable assets". *Econometrica.* 53 (4): 783–805, 1985.

[10] Spence, Michael, "Job Market Signaling," *Quarterly Journal of Economics*, vol. 87(3), pages 355-374, 1973.

[11] Stiglitz, Joseph E and Andrew Weiss, "Credit Rationing in Markets with Imperfect Information," *American Economic Review*, vol. 71(3), pages 393-410, 1981.

[12] Tirole, Jean, "Overcoming Adverse Selection: How Public Intervention Can Restore Market Functioning." *American Economic Review*, 102 (1): 29-59, 2012.

10 Appendix

Lemma 1 *If a real random variable $\tilde{X}$ has finite expectation then so does the random variable $\tilde{X}^+ = \max\{\tilde{X}, 0\}$.*

Proof. *Finite expectation means that both terms in the sum* $\int_{-\infty}^{0} x dF_{\tilde{X}}(x) + \int_{0}^{\infty} x dF_{\tilde{X}}(x)$ *are finite (where $F_{\tilde{X}}$ is the distribution function of $\tilde{X}$. The second term is just the expectation of $\tilde{X}^+$.* ■

Lemma 2 *At prices given by a fixed point of the function the market-wide average discounted expected lifetime utility of consumers, starting at any time period, is bounded above by a finite number independent of $\bar{p}$.*

Proof. By the previous lemma the mean of $\varepsilon_{hjt}^+ = \max\{\varepsilon_{hjt}, 0\}$ is finite. The utility flow in period t to a consumer h of type i from owning car k in class j is $u_{hkt} = v_{ijt} + q_{kt} + \varepsilon_{hjt}$. Because there are finitely many v_{ijt} and because $q_{kt} \in Q$ where Q is compact, there is a bound B independent of $\bar{p}$ such that, for all i, j, k, t, $u_{hkt} \leq B + \varepsilon_{hjt}^+ \leq B + \sum_{j \in J_t} \varepsilon_{hjt}^+$. The mean of the last sum is finite and thus we can place a uniform bound B' independent of $\bar{p}$ on the expected utility flow from owning cars. If the household buys a vehicle of class j and sells one of class j' in period t then its utility that period is bounded above by $B' + p_{jt} - p_{j't}$. Let $S(h, j, t)$ be an indicator function that, in period t, is equal to 1 if household h sells a car in class j, is equal to -1 if h buys a car in class j, and is equal to zero otherwise. We can bound household h utility in period

t above by $B' + \sum_{j \in J_t} S(h,j,t) p_{jt}$. Integrating over all h in the market, we see that average utility flow is bounded above by

$$B' + \sum_{j \in J_t} (S_{jt} - S_{jt}^E - D_{jt}) p_{jt}$$

and hence by $\sum_{j \in J_t} (S_{jt} - D_{jt}) p_{jt}$ the sum of prices multiplied by net excess supply. At a fixed point, if the excess supply of cars in class j is positive, then the associated price is at the minimum level $\underline{p}_j$. Hence the average expected utility derived in period t is bounded above by the product of the population and the maximum of the $\underline{p}_j$. Discounting and adding across periods this gives an upper bound, independent of $\bar{p}$, on expected discounted utility starting in any period. ∎