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ECONOMIC DISTRIBUTIONS AND PRIMITIVE DISTRIBUTIONS IN MONOPOLISTIC COMPETITION†

Abstract

We link fundamental technological and taste distributions to endogenous economic distributions of firm size (output, profit) and prices in extensions of canonical IO and Trade models. We develop a continuous logit model of monopolistic competition to show that exponential or normal distributions respectively generate Pareto or log-normal economic size distributions. Two groups of distributions (output, profit, and quality-cost; and price and cost) are linked through the technological relation between cost and quality-cost. We formulate a general monopolistic competition model and recover the demand structure, mark-ups, and the quality-cost distribution from the output and profit distributions. Adding the price distribution recovers the cost distribution and the relation between quality-cost and cost. We also find long-run equilibrium distributions as a function of the primitives. On the Trade side, we provide a parallel analysis for the CES and break the Pareto circle by introducing quality.

JEL Classification: L13

Keywords: ces, general monopolistic competition model, logit, pareto and log-normal distribution, price and profit dispersion and primitive and economic distributions

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1 Introduction

Distributions of economic variables have attracted the interest of economists at least since Pareto (1896). In industrial organization, firm size (output, sales, or profit) distributions have been analyzed, while different studies have looked at the distribution of prices within an industry. Firm sizes (profitability, say) within industries are wildly asymmetric, and frequently involve a long-tail of smaller firms. The idea of the long tail has recently been invoked prominently in studies of Internet Commerce (Anderson, 2006, Elberse and Oberholzer-Gee, 2006), and particular distributions – mainly the Pareto and log-normal – seem to fit the data well in other areas too (see Head, Mayer, and Thoenig, 2014).

In international trade, recent advances have enabled studying distributions of sales revenues (see, e.g., Eaton, Kortum, and Kramarz, 2011). The distributions of these "economic" variables are (presumably) jointly determined by the fundamental underlying distributions of tastes and technologies. In this paper we determine the links between the various distributions. We link the economic ones to each other and to the primitive distributions and tastes. Moreover, the primitives can be uncovered from the observed economic distributions.

To set the stage, we start by deploying the logit model of monopolistic competition, which we develop and extend here to a continuum of firms.[1] The logit is the workhorse model in structural empirical IO, and it readily incorporates taste and cost heterogeneity.[2] We show a three-way relation between two groups of distributions and the quality-to-cost relation: knowing one element from any two of these ties down the third. On one leg, we generate the relation between equilibrium profit dispersion, firm outputs, and the fundamental quality-cost distribution. On a second leg, we show the relation between the cost distribution and equilibrium

[1] An alternative tractable model we analyze is the logit's close cousin, the CES model.

[2] Ironically, Chamberlin (1933) is best remembered for his symmetric monopolistic competition analysis. Yet he went to great length to point out that he believed asymmetry to be the norm, and that symmetry was a very restrictive assumption. We model both quality and production cost differences across firms.

price dispersion. Knowing any one of the distributions on one leg suffices to determine the others on that leg. Moreover, knowing a distribution from each leg allows us to determine what the relation between cost and quality must be on the third leg. Some important equivalences include that normally distributed quality-costs induce log-normal distributions of profits, and that a power distribution of costs along with an exponential distribution of quality-costs leads to a Pareto distribution for profit. These results also apply to a long-run analysis in the spirit of Melitz (2003) with the set of active firms determined endogenously.

With that back-drop, we then broaden our scope by deploying a more general model of monopolistic competition by relaxing the constant mark-up property of the logit. We first show how the demand function delivers a mark-up function, and then we show our key converse result that the mark-up (or "pass-through" function of Weyl and Fabinger, 2013) determines the form of the demand function. We next engage these results to show how the economic profit and output distributions allow us to determine the demand function and quality-cost distributions. Knowledge of the price distribution then enables us to recover the other primitives, which are the cost distribution and the relation between costs and quality-cost.

We provide a parallel analysis for the CES model.[3] The CES representative consumer model is widely used in economics in conjunction with a market structure assumption of monopolistic competition It is used as a theoretical component in the New Economic Geography and Urban Economics, it is the linchpin of Endogenous Growth Theory, Keynesian underpinnings in Macro, and of course, Industrial Organization. The current most intensive use of the model is in International Trade, following Melitz (2003), where it is at the heart of empirical estimation. The convenience of the model stems from its analytic manipulability. The CES model delivers equilibrium mark-ups proportional to marginal costs, and so delivers market imperfection (im-

[3]The Logit is an attractive alternative framework to the CES. Anderson, de Palma, and Thisse (1992) have shown that the CES can be viewed as a form of Logit model.

perfect competition) in a simple way without complex market interaction. The standard models in this vein (following Melitz, 2003) assume that firms' unit production costs are heterogeneous.

However, when we apply this model to distributions, if one distribution (such as profits) is described as a Pareto distribution, then the distributions of all the economic variables lie in the Pareto family. This we call the "Pareto circle" (or, more generally, the CES circle because the result applies to any distribution). The circle is broken by introducing qualities (as do Baldwin and Harrigan, 2011, and Feenstra and Romalis, 2014) into the demand model. Doing so delivers two fundamental drivers of equilibrium distributions (instead of just one) – the cost distribution and the quality/cost one. Even if one distribution is Pareto, then others can take different forms. Most notably, the output distribution depends on the cost distribution (as before) but now also on the quality/cost distribution.

2 The Logit model of monopolistic competition

There is a continuum of active firms. Each firm, i, is associated to a distinctive quality v_i, (constant) marginal production cost, c_i, and chooses a price, p_i.[4] Let Ω be the set of active (producing) firms, and let ω denote an element of this set. Total demand is normalized to 1, w.l.o.g. Demand for Firm i is a Logit function of active firms' qualities and prices:[5]

$$y_i = \frac{\exp\left(\frac{v_i - p_i}{\mu}\right)}{\int_{\omega\in\Omega} \exp\left(\frac{v(\omega)-p(\omega)}{\mu}\right) d\omega + \exp\left(\frac{v_0}{\mu}\right)}, \quad i \in \Omega, \tag{1}$$

where $\mu > 0$ measures the degree of product heterogeneity and $v_0 \in (-\infty, \infty)$ measures the attractivity of the outside option (which could also represent a competitive sector). We thus adapt the continuous Logit model (see Ben-Akiva and Watanada, 1981) to monopolistic competition.[6]

[4] Both v_i and c_i can be optimally determined by the firm, according to some fundamental firm "productivity." More details anon.

[5] We assume that the integral in the denominator is bounded: conditions are given below.

[6] Anderson et al. (1992) show that logit demands can be generated from an entropic representative consumer utility function as well as the traditional discrete choice theoretic root (see McFadden, 1978).

Consumer choices are driven by two forces. First, absent product differentiation, consumers want the best quality-price deal (highest $v_i - p_i$). Second, consumers have idiosyncratic tastes for differentiated products. When product differentiation (measured by μ) is very large, quality-price is unimportant and each good has the *same* purchase probability. Otherwise, there is a trade-off between objective quality (vertical differentiation) and subjective quality (horizontal differentiation). The (gross) profit for Firm i is $\pi_i = (p_i - c_i)\, y_i$, $i \in \Omega$. Because the firm has no impact on the denominator in (1), under monopolistic competition the own-demand derivative is $\frac{dy_i}{dp_i} = \frac{-y_i}{\mu}$, $i \in \Omega$. Hence $\frac{d\pi_i}{dp_i} = y_i \left[1 - \frac{(p_i - c_i)}{\mu}\right]$, $i \in \Omega$, and, since the term inside the square brackets is strictly decreasing in p_i, the profit function is strictly quasi-concave and the profit-maximizing price of Firm i is[7]

$$p_i^* = c_i + \mu, \qquad i \in \Omega. \tag{2}$$

The absolute mark-up is the same for all firms.[8] The corresponding equilibrium outputs are

$$y_i = \frac{\exp\left(\frac{v_i - c_i}{\mu}\right)}{\int_{\omega \in \Omega} \exp\left(\frac{v(\omega) - c(\omega)}{\mu}\right) d\omega + \mathcal{V}_0}, \qquad i \in \Omega, \tag{3}$$

where $\mathcal{V}_0 \equiv \exp\left(\frac{v_0}{\mu} + 1\right) \geq 0$ (and where y_i henceforth stands for the equilibrium output). Let $x = v - c$ be a one-dimensional parameterization of quality-cost (to be read as quality minus cost). (3) indicates an output ranking over firms:

$$y_i > y_j \quad \text{if and only if } x_i > x_j, \qquad i, j \in \Omega.$$

The equilibrium (gross) profit is μy_i, $i \in \Omega$, so outputs and profits are fully characterized

[7] For oligopoly with n firms, the equilibrium prices are (implicit) solutions to $p_i^* = c_i + \frac{\mu}{1 - y_i}$, $i = 1...n$. Under symmetry, $p^* = c + \frac{\mu n}{n-1}$, which converges to $c + \mu$ as $n \to \infty$ (Anderson, de Palma, and Thisse, 1992, Ch.7).

[8] The CES model gives a constant *relative* mark-up property, $p_i^* = c_i\,(1 + \mu)$, regardless of quality (see Section 7). The similarity between the Logit and CES is not fortuitous: μ is related to ρ in CES models by $\mu = \frac{1-\rho}{\rho}$. Both models can be construed as sharing their individual discrete choice roots (Anderson et al. 1992).

by quality-cost levels:

Proposition 1 *In the Logit Monopolistic Competition model, all firms set the same absolute mark-up, μ. Higher quality-cost entails higher equilibrium output and profit.*

A high quality with a high cost is equivalent (for output and profit) to a low quality/ low cost combination. Hence, all we need to track is the distribution of quality-cost. Insofar as higher qualities are also higher costs in practice, then they are also higher priced. However, output and profitability may well be highest for medium-quality products (see Section 4.2).

2.1 Quality-cost, output, and profit distributions

Let the distribution of quality-cost be $F_X(x) = \Pr(X < x)$, with density $f_X(\cdot)$ and support $[\underline{x}, \infty)$. We seek the corresponding distribution of equilibrium output, $F_Y(y)$, and the relation between x and y is[9]

$$y = \frac{\exp(x/\mu)}{D}, \quad y \geq \underline{y} = \frac{\exp(\underline{x}/\mu)}{D}, \tag{4}$$

where we assume henceforth that $f_X(\cdot)$ ensures the output denominator D is finite:[10]

$$D = M \int_{u \geq \underline{x}} \exp(u/\mu) f_X(u)\, du + \mathcal{V}_0, \tag{5}$$

and $M = \|\Omega\|$ is the total mass of firms. Equilibrium (gross) profit is $\pi = \mu y \geq \underline{\pi} = \mu \underline{y}$. Let y_{av} and π_{av} denote average output and profit, respectively.

Theorem 1 *For the Logit Monopolistic Competition model, the quality-cost distribution, $F_X(x)$, generates the equilibrium output distribution $F_Y(y) = F_X(\mu \ln(yD))$ and the equilibrium profit distribution $F_\Pi(\pi) = F_X(\mu \ln(\pi D/\mu))$, where D is given by (5). Conversely, $F_X(x)$ can be derived from the equilibrium output distribution as $F_Y\left(\frac{\exp(x/\mu)}{D_y}\right)$, where $D_y = \mathcal{V}_0/(1 - M y_{av})$; or from the equilibrium profit distribution as $F_\Pi\left(\mu \frac{\exp(x/\mu)}{D_\pi}\right)$, where $D_\pi = \mathcal{V}_0/\left(1 - \frac{M}{\mu}\pi_{av}\right)$.*

[9] Here all firms are active. Section 6 introduces fixed costs to determine endogenously the set of active firms.
[10] This holds true for any finite support as well as for the examples below.

(Proof in Appendix 2). The key relation underlying the twinning of distributions is the increasing relation between quality-cost and output (or profit) for the Logit. Similar increasing relations hold for other models like the model in Section 5 below, as well as the CES (under some restrictions: see Section 7). The theorem uses this increasing relation to describe how quality-costs can be determined from output or profit distributions. A specific quality-cost distribution generates a specific output (resp. profit) distribution. Conversely, this output or profit distribution could only have been generated from the initial quality-cost distribution.

2.2 Comparative statics of distributions

We here briefly consider the comparative static properties. Because we are dealing with distributions, the natural way of doing so is to engage first order stochastic dominance (fosd).

Proposition 2 *A fosd increase or a mean-preserving spread in quality-cost increases mean output and mean profit, and strictly so if the market is not fully covered (i.e., if $\mathcal{V}_0 > 0$).*

Even though the proof of the first part of the Proposition is straightforward, it belies some counteracting effects. While moving up quality-cost mass will move up output mass ceteris paribus, it also increases competition for all the other firms (a D effect), which ceteris paribus reduces their output. Mean output does not necessarily rise if mean quality-cost rises.[11]

Because the relation between output and profit distributions ($F_{\Pi}(\pi) = F_Y(\pi/\mu)$) does not involve D, a fosd increase in output implies an increase in profit, and vice versa. However, a fosd increase in quality-cost does not necessarily lead to a fosd increase in output. Suppose for example that the increase in quality-cost is small for low quality-costs, but large for high ones. Then competition is intensified (an increase in D), and output at the bottom end goes down, while rising at the top end. So then there can be a rotation of $F_Y(.)$ (in the sense of

[11] Mean output rises with a mean-preserving spread (part (ii)) but then if the mean of quality-cost is reduced slightly, mean output can still go up overall, so the two means can move in opposite directions.

Johnson and Myatt, 2006) without fosd (a similar rotation is delivered in Proposition 3 below). Nevertheless, specific examples do deliver stronger relations, as we show in Section 3.

We next determine how the economic parameters feed through into the endogenous economic distributions. To do so, we first prove an ancillary result of independent economic interest.

Lemma 1 *For the logit model, consumer surplus (at fixed prices) increases with product differentiation, μ; hence $\mu \ln D$ increases in μ.*

The proof proceeds by showing that the derivative of consumer surplus is given by the Shannon (1948) measure of information (entropy), which is positive.

Proposition 3 *A more attractive outside option ($\mathcal{V}_0$) fosd decreases outputs and profits. More product differentiation (μ) fosd increases outputs and profits for low quality-cost and fosd decreases them for high quality-cost; a lower profit implies a firm has a lower output.*

The first result is quite obvious, but the impact of higher product heterogeneity is more subtle. When μ goes up, weak (low quality-cost) firms are helped and good ones are hurt. The intuition is as follows. With little product differentiation, consumers tend to buy the best quality-cost products. With more product differentiation (which increases the mark-up), consumers tend to buy more of the low quality-cost goods (which have lower outputs, as per Proposition 1) and less of the high quality-cost goods (which have higher outputs). Hence, higher μ evens out demands across options. The fact that output may decrease and profit increase with μ follows because $\pi_i = \mu y_i$. Thus it can happen that doubling μ does not double the profit of the top quality-cost firms, but it may more than double the profit of the lowest quality-cost firms. Whether high or low qualities are most profitable depends on whether quality-costs rise or fall with quality.

3 Specific distributions

We derive the equilibrium profit distributions: the equilibrium output distributions are analogous (because $\pi = \mu y$). We express all examples from quality-cost distributions to implied profit distributions: from Theorem 1 the reverse relations hold too, and output relations are analogous. Proofs are in Appendix 2.

The *normal* distribution is perhaps the most natural primitive assumption to take for quality-costs. Then profit $\Pi \in (0, \infty)$ is log-normally distributed. The log-normal has sometimes been fitted to firm size distribution (see Cabral and Mata, 2003, for a well-cited study of Portuguese firms). Note that a truncated normal begets a truncated log-normal (which is therefore important once we consider free entry equilibria below).

The simplest text-book case is the *uniform* distribution. Then the equilibrium profit Π has distribution $F_{\Pi}(\pi) = \mu \ln\left(\frac{\pi D}{\mu}\right)$ and its density is unit elastic. A *truncated Pareto* distribution leads to a truncated Log-Pareto for profit (or output).

At a simplistic level, Theorem 1 indicates that we just need to find the log-distribution of the seed distribution. However, we still need to match parameters, as done in Appendix 2 for the examples, and we also need to find the corresponding expression for D and ensure it is defined. Notice too that the methods described above work for more general demands under monopolistic competition (see Section 5).

The most successful function to fit the distribution of firm size has been the Pareto. We reverse-engineer using Theorem 1 to find the distribution of quality-cost. This gives:

Proposition 4 *Let quality-cost be exponentially distributed:* $F_X(x) = 1 - \exp(-\lambda(x - \underline{x}))$, $\lambda > 0$, $\underline{x} > 0$, $x \in [\underline{x}, \infty)$, *with* $\lambda\mu > 1$. *Then equilibrium profit is Pareto distributed:* $F_{\Pi}(\pi) = 1 - \left(\frac{\underline{\pi}}{\pi}\right)^{\alpha_\pi}$, *where* $\alpha_\pi = \lambda\mu > 1$. *Equilibrium output is Pareto distributed:* $F_Y(y) = 1 - \left(\frac{\underline{y}}{y}\right)^{\alpha_y}$, *where* $\alpha_y = \lambda\mu > 1$. *A Pareto distribution for equilibrium output or profit can only be generated* *by an exponential distribution of quality-costs.*

Thus the shape parameter, $\alpha_y = \alpha_\pi$, for the endogenous economic distributions depends just on the product of the taste heterogeneity and the technology shape parameter. D is bounded if $\mu > 1/\lambda$: which requires that taste heterogeneity exceeds average quality-cost.

4 Three-way synthesis

The distributions of quality-cost, output, and profit are determined from any one of them (Theorem 1). Likewise, price and cost distributions are determined from either one. The link between any of the former distributions and either of the latter two is determined by the relation between costs and quality-costs. This section draws together these relations, and shows how the link between distributions can be determined. Conversely, knowing the relation between cost and quality-cost and one distribution enables us to tie down the other distributions.

Note some special case results. First, there is no price dispersion if and only if there is no cost dispersion. Second, there is no profit dispersion if and only if there is no quality-cost dispersion: then there is only cost dispersion, with price dispersion mirroring the cost dispersion as per the formula in 1 below. The "classic" symmetry assumption often analyzed in the literature (e.g., Chamberlin, 1933) has neither cost nor profit dispersion.

4.1 Legs and Bridge

We proceed by describing the two separate groups of distributions (the two "legs") and how they are linked (the "bridge").

Leg #1: Quality-cost, output, and profit. As shown in Theorem 1, knowledge of any one of these distributions ties down the other two.

Leg #2: Prices and costs. The distribution of costs $F_C(c)$ and the distribution of prices $F_P(p)$ are related by the shift, $P = C + \mu$, so $F_C(c) = \Pr(P < c + \mu) = F_P(c + \mu)$, with

$\underline{p} = \underline{c} + \mu$. Conversely, knowing the price distribution ties down the cost distribution. If the price distribution follows the Pareto form (suggested as empirically viable) $F_P(p) = 1 - \left(\frac{\underline{p}}{p}\right)^{\alpha_p}$, with $p \in [\underline{p}, \infty)$ and $\alpha_p > 1$, the corresponding cost distribution is

$$F_C(c) = 1 - \left(\frac{\underline{c} + \mu}{c + \mu}\right)^{\alpha_p}, \quad c \in [\underline{c}, \infty), \ \alpha_p > 1. \tag{6}$$

Bridge: Quality-costs and costs. We just showed that knowing one distribution from either leg enables us to determine the other(s) on that leg. We link the distributions across the two legs by postulating a functional relation between tastes and technology, and so we link quality-cost from the first leg with cost from the second leg.

Suppose then that it is known that $x = \beta(c)$. Then we can determine the relation between quality, v, and cost as $v = \beta(c) + c$. Several cases are possible. Normally, one might expect that quality should increase with cost, so $\beta'(c) > -1$. Otherwise, quality might be increasing or decreasing in c or non-monotonic. A hump-shaped relation represents highest quality-costs for intermediate cost levels (see Figure 2 below).

We can either treat $\beta(c)$ as a datum to determine other relations, or else we can infer it from a seed distribution from each distribution leg. In the sequel, we analyze $\beta'(c) > 0$, i.e., "better" products have higher costs. Other cases are treated after the main analysis. The material that follows that pertains to $\beta(c)$ applies also to the more general model of Section 5.

We can also view both x and c as determined by the firm, depending on the firm's type. In this case the bridge function traces the relation between optimal choices.[12]

[12] Because higher x gives higher equilibrium profit, then each firm wants to maximize x under whatever technological transformation it faces. For example, following the lines of Feenstra and Romalis (2014), we could assume a production function of the form $v = l^\alpha \theta$ with corresponding cost wl, where l is labor input, w is the wage, $\alpha \in (0,1)$, and θ is a firm-specific productivity parameter. Then we get a linear bridge function $x = \frac{1-\alpha}{\alpha} c$. This one shows up in the last example in Section 5.

4.2 Synthesis

We now synthesize the relations between the different groups of relations. We assume throughout that the logit denominator D is finite (which holds, e.g., if distributions are bounded).

Theorem 2 *For the Logit monopolistic competition model, if one element is known from two of the three following groups, then all elements are known:*

i) a distribution of quality-cost, profit, output (F_X,F_Π,F_Y);

ii) a distribution of price or cost (F_P,F_C);

iii) an increasing relation between any pair of x, v, and c.

The construction of $F_C(c)$ from $F_X(x)$ and $\beta(c)$ is shown in Figure 1. In the upper right panel we have the "seed" distribution $F_X(x)$, and below it is $\beta^{-1}(x)$. Values of x map into values of c via the relation $\beta^{-1}(x)$ in the lower right panel and hence through the lower left panel into values of c in the upper left panel, where the corresponding value from $F_X(x)$ therefore yields the desired value of $F_C(c)$. The Figure also shows the converse constructions.

INSERT FIGURE 1: Relation between (increasing) cost and quality-cost.

Example Increasing $\beta(c)$ with Pareto distributed profits and prices

$F_X(x)$ is exponential and given in Proposition 4. Because $\beta(\cdot)$ is increasing,

$$F_C(c) = F_X(\beta(c)) = 1 - \exp[-\lambda(\beta(c) - \beta(\underline{c}))], \ \lambda = \frac{\alpha_\pi}{\mu} > 0, \ \underline{c} > 0, \ c \in [\underline{c}, \infty).$$

A Pareto price distribution with shape parameter α_p delivers the cost distribution (6). Equating these two expressions gives

$$\beta(c) = \beta(\underline{c}) + \mu\frac{\alpha_p}{\alpha_\pi}\ln\left(\frac{c+\mu}{\underline{c}+\mu}\right), \ \beta(c) \in [\beta(\underline{c}), \infty].$$

Thus $\beta'(c) > 0$ so that valuations rise faster than costs. The lower bound of the distribution $\beta(\underline{c}) = \underline{x}$ is given by Proposition 4 and is given in the next Proposition:

Proposition 5 *Let $F_P(p)$ be Pareto distributed with shape parameter α_p and let $F_\Pi(\pi)$ be Pareto distributed with shape parameter $\alpha_\pi > 1$ and $\underline{\pi} < \frac{\mu}{M}\frac{\alpha_\pi - 1}{\alpha_\pi}$, and suppose that $x = \beta(c)$ is an increasing function. Then $\beta(c) = \beta(\underline{c}) + \mu\frac{\alpha_p}{\alpha_\pi}\ln\left(\frac{c+\mu}{\underline{c}+\mu}\right)$, where $\beta(\underline{c}) = -\mu\ln\left[\frac{1}{\mathcal{V}_0}\left(\frac{\mu}{\underline{\pi}} - \frac{\alpha_\pi M}{\alpha_\pi - 1}\right)\right]$.*

Figure 1 illustrates (parameters are $\alpha_p = \alpha_\pi = \mu = 2$, $\beta(\underline{c}) = 0$, $\underline{c} = 0$, and $\underline{x} = 1$).

Now consider when the higher quality-cost products are at the lower end of the cost spectrum. This is an important case because it corresponds to the extant literature à la Melitz (2003), which entertains only cost differences.[13] Quality-costs decrease with cost, which entails a reversal of the ordering of products. The basic problem though is the same, and so analogous results to those above hold. To construct the function $F_X(x)$ from $F_C(c)$ and given $\beta(c)$ decreasing on the relevant support $[\underline{c}, \bar{c}]$, we use the relation $F_X(x) = \Pr(\beta(C) < x) = \Pr\left(C > \beta^{-1}(x)\right) = 1 - F_C\left(\beta^{-1}(x)\right)$. Conversely, a decreasing β function can be constructed from $F_C(c)$ and $F_X(x)$.

We now apply this analysis to the case of cost heterogeneity alone (which parallels our later analysis for the CES). Let v be constant, and write $\beta(c) = \bar{v} - c$. Then

$$F_X(x) = \Pr(\bar{v} - C < x) = \Pr(C > \bar{v} - x) = 1 - F_C(\bar{v} - x).$$

Suppose for illustration that prices are Pareto distributed so that $F_C(c)$ is given by (6). Hence we get the power distribution

$$F_X(x) = \left(\frac{\bar{v} - \underline{x} + \mu}{\bar{v} - x + \mu}\right)^{\alpha_p}, \qquad x \in (-\infty, \underline{x}].$$

We next consider the case where $\beta(c)$ is increasing from $\underline{c}$ to $\hat{c}$ and decreasing from $\hat{c}$ to $\bar{c}$. Quality rises faster than cost at first, and then rises slower or even falls (if $\beta' < -1$). This case involves highest quality-cost (and hence highest output and profit, by Theorem 1) for middling

[13] We situate this on its home ground in the CES model in Section 7 (Section 6 considers the endogenous set of product quality-costs for logit).

cost levels. The cumulative quality-cost distribution is derived from the two pieces. Suppose that $\beta(c) < \beta(\bar{c})$ for $c \in [\underline{c}, \tilde{c})$ (and so $\beta(\tilde{c}) = \beta(\bar{c})$). Then $F_X(x)$ is derived from $F_C(c)$ via $F_X(x) = F_C\left(\beta^{-1}(x)\right)$ for $x \in [\beta(\underline{c}), \beta(\tilde{c}))$. Higher x values can come from either the increasing or decreasing part of β, and we need to sum the two contributions.

Define $\beta_U^{-1}(x)$ as the inverse function for β increasing (i.e., corresponding to $c < \hat{c}$) and $\beta_D^{-1}(x)$ as the inverse function for β decreasing (i.e., corresponding to $c > \hat{c}$). Then for $x \in [\beta(\tilde{c}), \beta(\hat{c})]$, $F_X(x)$ is given as the sum of the contributions from the two parts, as per the statement in the second line below. Summarizing:

$$\begin{aligned} F_X(x) &= F_C\left(\beta^{-1}(x)\right) && \text{for } x \in [\beta(\underline{c}), \beta(\tilde{c})) \\ F_X(x) &= F_C\left(\beta_U^{-1}(x)\right) + 1 - F_C\left(\beta_D^{-1}(x)\right) && \text{for } x \in [\beta(\tilde{c}), \beta(\hat{c})] \end{aligned}$$

(notice indeed that $F_X(x)$ is increasing, with a kink up at $\beta(\tilde{c})$, and that $F_X(\beta(\hat{c})) = 1$). The $\beta(.)$ function used above is illustrated in Figure 2, where $F_C(c) = c$ for $c \in [0, 1]$, and $\beta(c) = c\left(\frac{4}{3} - c\right)$ for $c \in [0, 1]$, so that $\tilde{c} = \frac{1}{3}$ and $\hat{c} = \frac{2}{3}$. Then $F_X(x) = \frac{2}{3} - \left(\frac{4}{9} - x\right)^{1/2}$ for $x \in [0, \frac{1}{3}]$ and $F_X(x) = 1 - 2\left(\frac{4}{9} - x\right)^{1/2}$ for $x \in [\frac{1}{3}, \frac{4}{9}]$.

INSERT FIGURE 2: Hump relation between cost and quality-cost.

5 Recovering demand from economic distributions

The logit set-up has each firm effectively facing a monopoly problem where the price choice is independent of the actions of rivals. In this spirit, we now model demand in the monopolistic competition model through firms facing individual demand curves with optimal prices that are independent of the aggregate value. We dispense with the logit model property that the mark-up is constant (which we have so far taken as a datum in developing the links between the various distributions), and now explicitly allow for endogenous type-dependent mark-ups.

This section delivers what are perhaps the strongest results of the paper. We allow for a general demand formulation as an additional primitive to the model, and show how the primitives feed through to the endogenous economic distributions and variables. Conversely, the derived economic distributions can be reverse engineered to back out the model's primitives.

We first give the demand model, and derive the equilibrium mark-up schedule in Theorem 3 as a function of firm quality-cost (x). Theorem 4 inverts the mark-ups to deliver not only the equilibrium output choices, but also the form of the demand curve (on the support corresponding to the set of quality-costs in the market). This analysis constitutes an stand-alone contribution to the theory of monopoly pass-through, extending Weyl and Fabinger (2013) by working from pass-through *back* to implied demands.

Theorem 5 shows how the (potentially observable) output and profit distributions can be inverted to determine the underlying primitive distribution of quality-cost, $F_X(x)$, and the underlying demand form. This step also determines the equilibrium mark-up distribution, which in turn determines (Theorem 6) the underlying distribution of costs, $F_C(c)$, from the (potentially observable) economic distribution of prices, $F_P(p)$. This last step also enables us to uncover the "bridge" function, $\beta(c)$, from $F_C(c)$ and $F_X(x)$. Throughout, we assume the appropriate monotonicity conditions that ensure invertibility (see the analogous discussion in Section 4.1).

5.1 Demand and mark-ups

Suppose now that demands are generated from the relation

$$y_i = \frac{h(v_i - p_i)}{D}, \quad i \in \Omega, \tag{7}$$

which generalizes (1), and where D is an aggregate value determined by the actions of all firms, but treated as constant by firms under the monopolistic competition assumption. We refer to $h(.)$ as a "quasi-demand." It is a positive, increasing, twice differentiable function of quality-price, and strictly (-1)-concave.[14] The exponential form of the Logit is a log-linear quasi-demand: other cases are spotlighted below.

One useful case to think about the quasi-demand is as a "scale value" in a Lucian demand system (based on the IIA property of the Logit, which property is shared with the CES), where $D = \int_{\omega\in\Omega} h(v(\omega) - p(\omega))\, d\omega + \mathcal{V}_0$, or

$$D = \frac{\mathcal{V}_0}{1 - M\int_{x>\underline{x}} y(x)\, f_X(x)\, dx}. \tag{8}$$

There is leeway for normalization here. $\mathcal{V}_0$ can be set to one, or else the equilibrium h for the lowest quality-cost firm can be one (this is $h^*(\underline{x})$ below).[15] We will follow the second route, and express quasi-demands relative to the base value. Hence, when we say below (e.g., in Theorem 4) that $h(.)$ is uniquely determined, it means up to a positive multiplicative factor. As seen below, we can also set $\underline{x} = 0$ and rescale how we measure quality-cost.

We now return to the more general case for D. Firm i's profit is $\pi_i = (p_i - c_i)\frac{h(v_i - p_i)}{D} = m_i \frac{h(x_i - m_i)}{D}$, $i \in \Omega$, where $m_i = p_i - c_i$ is i's mark-up.[16] Under monopolistic competition, the equilibrium mark-up satisfies

$$m_i = \frac{h(x_i - m_i)}{h'(x_i - m_i)}, \qquad i \in \Omega. \tag{9}$$

Theorem 3 *Let $h(.)$ be a positive, increasing, strictly (-1)-concave, and twice differentiable quasi-demand function. Then the associated monopolistic competition mark-up, $\mu(x)$ is the unique solution to (9), with $\mu'(x) < 1$. The associated equilibrium quasi-demand, $h^*(x) \equiv$*

[14] This is equivalent to $\frac{1}{h(.)}$ strictly convex, and is the minimal condition ensuring a maximum to profit. See Caplin and Nalebuff (1991) and Anderson, de Palma, and Thisse (1992) for more on ρ-concave functions; and Weyl and Fabinger (2013) for the properties of pass-through as a function of demand curvature: the analogue to cost pass-through is here the complementary feature of quality-pass-through.

[15] The first route is most familiar to econometricians in discrete choice. The second route does not allow us to deal nicely with endogenous quality-costs (which we do in the long-run model of the next Section).

[16] By the envelope theorem, the maximized value, $\pi_i^*(x_i)$ is increasing in x_i: see also Theorem 3.

$h(x-\mu(x))$, is strictly increasing, as is $\hat{\pi}^*(x) = \mu(x) h^*(x)$.

Proof. The solution to (9), denoted $\mu(x)$, is uniquely determined (and strictly positive) when the RHS of (9) has slope less than one, as is implied by $h(.)$ being strictly (-1)-concave. Applying the implicit function theorem to (9) shows that

$$\mu'(x) = \frac{\left(\frac{h(x-m)}{h'(x-m)}\right)'}{1+\left(\frac{h(x-m)}{h'(x-m)}\right)'} < 1, \tag{10}$$

where the numerator is strictly positive when h is (-1)-concave.[17] The mark-up thus has slope less than one. Let $h^*(x) = h(x-\mu(x))$ denote the value of $h(.)$ under the profit-maximizing mark-up. Then, given that $\mu'(x) < 1$, $h^*(x)$ is strictly increasing, as claimed, because

$$dh^*(x)/dx = (1-\mu'(x))\, h'(x-\mu(x)) > 0. \tag{11}$$

Finally, $\hat{\pi}^*(x) = \mu(x) h^*(x)$ is strictly increasing from the envelope theorem. ■

The only quasi-demand function with constant mark-up is the exponential (associated to the Logit), which has $h(\cdot)$ log-linear, and so $\frac{h(x-m)}{h'(x-m)}$ is constant. For $h(\cdot)$ strictly log-concave, $\mu'(x) > 0$, so firms with higher quality-costs have higher mark-ups in the cross-section of firm types. They also have higher equilibrium outputs. When $h(\cdot)$ is strictly log-convex, the mark-up decreases with x: this is analogous to a cost pass-through greater than 1. Notice that the property $\mu'(x) < 1$ is just the property that price never goes down as costs increase.

An important special case is when quasi-demand is $\rho-$linear (which means that h^ρ is linear). Suppose then that $h(.) = (v+k-p)^{1/\rho}$, where k is a constant. Then

$$\mu(x) = \frac{\rho(x+k)}{1+\rho},$$

which is linear in x. For $\rho = 1$ quasi-demand is linear and the standard property is apparent

[17] We can have quality rise and mark-up go down immensely near the -1-concave limit: think too of cost pass-through; with a demand 1/p then a zero cost gives a price of zero, but a small cost gives an infinite price.

that mark-ups rise fifty cents on the dollar with quality-cost.[18] Note for such $\rho-$linear demands that $h^*(x) = \left(\frac{x+k}{1+\rho}\right)^{1/\rho}$ and it is readily verified that $\frac{dh^*(x)/dx}{h^*(x)} = \frac{1}{\rho(x+k)} = \frac{1-\mu'(x)}{\mu(x)}$ (see (12) below).

The converse result to Theorem 3 indicates how the mark-up function can be inverted to determine the form of h^* (and hence $h(.)$).

Theorem 4 *Let there be a mark-up function $\mu(x)$ for $x \in [\underline{x}, \bar{x}]$ with $\mu'(x) < 1$. Then there exists a unique equilibrium quasi-demand function $h^*(\cdot)$ defined on its support $[\underline{x}, \bar{x}]$ and given by (13). The associated primitive quasi-demand function $h(\cdot)$, given by (14), is strictly (-1)-concave on its support $[\underline{x} - \mu(\underline{x}), \bar{x} - \mu(\bar{x})]$.*

Proof. First note from (9) and (11) that

$$\frac{dh^*(x)/dx}{h^*(x)} = \frac{(1-\mu'(x))\,h'(x-\mu(x))}{h(x-\mu(x))} = \frac{(1-\mu'(x))}{\mu(x)} \equiv g(x). \tag{12}$$

Thus $[\ln h^*(x)]' = g(x)$, and so $\ln\left(\frac{h^*(x)}{h^*(\underline{x})}\right) = \int_{\underline{x}}^{x} g(v)\,dv$, which implies

$$h^*(x) = h^*(\underline{x}) \exp\left(\int_{\underline{x}}^{x} g(v)\,dv\right), \quad x \geq \underline{x}, \tag{13}$$

which therefore determines $h^*(x)$ up to a positive factor.

We can now use $h^*(x)$ to back out the original function $h(x-m)$ via the following steps. First, define $u = \phi(x) = x - \mu(x)$, which is monotone increasing because $1 - \mu'(x) > 0$, so the inverse function $\phi^{-1}(\cdot)$ is increasing. Now, $h(u) = h^*\left(\phi^{-1}(u)\right)$ with $u \in [\underline{x} - \mu(\underline{x}), \bar{x} - \mu(\bar{x})]$ and thus the function $h(\cdot)$ is recovered on the support. Using (13) with $h(u) = h^*\left(\phi^{-1}(u)\right)$,

$$h(u) = h^*(\underline{x}) \exp \int_{\underline{x}}^{\phi^{-1}(u)} g(v)\,dv, \tag{14}$$

[18] If $\rho = 0$ we have log-linearity. The astute reader will note that the expression given does not go to the constant mark-up of the Logit. To properly derive the Logit limit, we could write instead our ρ-linear demand as $h(.) = a\left(1 + \frac{(v+k-p)}{\theta}\rho\right)^{1/\rho}$ which has the limit $\lim_{\rho \to 0} h(.) = a \exp\left(\frac{(v+k-p)}{\theta}\right)$.

and so (by (12)), and because $\phi'(x) = 1 - \mu'(x)$:

$$\frac{h(u)}{h'(u)} = \frac{1}{g\left(\phi^{-1}(u)\right)\left[\phi^{-1}(u)\right]'} = \frac{\phi'(x)}{\frac{(1-\mu'(x))}{\mu(x)}} = \mu(x).$$

Thus $h(u)$ is strictly (-1)-concave because[19]

$$\left[\frac{h(u)}{h'(u)}\right]' = \frac{\mu'(x)}{1-\mu'(x)} > -1.$$

■

The steps above are readily confirmed for the $\rho-$linear example given before Theorem 4. Taking Theorems 3 and 4 together, knowing either $\mu(x)$ or $h(.)$ suffices to determine the other and $h^*(x)$. This constitutes a strong characterization result for monopoly pass-through (see Weyl and Fabinger, 2013, for the state of the art, which deeply engages ρ-concave functions).

Notice that the function $h(\cdot)$ is tied down only on the support corresponding to the domain on which we have information about the equilibrium value in the market. Outside that support, we know only that $h(\cdot)$ must be consistent with the maximizer $\mu(x)$, which restricts the shape of $h(\cdot)$ to be not "too" convex. The case of positive quality pass-through (which is equivalent to cost pass-through below 100%) is associated to log-concave demand.

5.2 Deriving demand form from output and profit distributions

We here engage the output and profit distributions (F_Y and F_Π) to show how to back out the underlying quality-cost distribution (F_X), and the implied demand. Before this reverse engineering, we first determine how the primitives F_X and $h(.)$ generate the pertinent economic distributions and mark-ups.

[19] When $h(u)$ is strictly (-1)-concave, then $h(u)h''(u) - 2[h'(u)]^2 < 0$, which rearranges to $\left[\frac{h(u)}{h'(u)}\right]' > -1$.

As shown already, $h^*(x)$ and $\mu(x)$ are derived from $h(.)$ via Theorems 3 and 4. Now note

$$F_Y(y) = \Pr\left(Y < \frac{h^*(x)}{D}\right) = \Pr\left(DY < h^*(x)\right) = F_X\left(h^{*-1}(DY)\right),$$

and, analogously,

$$F_\Pi(\pi) = \Pr\left(\Pi < \frac{\hat{\pi}^*(x)}{D}\right) = \Pr\left(D\Pi < \hat{\pi}^*(x)\right) = F_X\left(\hat{\pi}^{*-1}(D\Pi)\right),$$

where we have used Theorem 3 that $h^*(x)$ and $\hat{\pi}^*(x)$ are strictly increasing.

The converse result is the key theorem. It tells us how to uncover the primitives from the economic distributions F_Y and F_Π.

Theorem 5 *Consider a demand model (7) under monopolistic competition. Assume that the corresponding distributions of output, F_Y, and profit, F_Π, are known. Then the quality-cost distribution, F_X, is given by (19) below, the mark-up function is found from (17), and equilibrium quasi-demand is found from (16).*

Proof. We know that $h^*(x)$ and hence $y = \frac{h^*(x)}{D}$ are strictly increasing in x, and so too is $\hat{\pi}^*(x) = \mu(x)\, h^*(x)$ (by Theorem 3). We hence choose some arbitrary level $z \in (0,1)$ such that

$$F_X(x) = F_Y(y) = F_\Pi(\pi) = z. \tag{15}$$

This means that all firm types with quality-cost levels below $x = F_X^{-1}(z)$ are the firms with outputs and profits below y and π. For this proof, we introduce z as an argument into the various outcome variables to track the dependence of the variables on the level of $z(x)$. Then we can write $y(z) = F_Y^{-1}(z)$ and quasi-demand is

$$h^*(x) = Dy(z(x)) = DF_Y^{-1}(F_X(x)). \tag{16}$$

Because $\pi^*(z) = m(z)\, y(z) = F_\Pi^{-1}(z)$ then

$$m(z) = \frac{F_\Pi^{-1}(z(x))}{F_Y^{-1}(z(x))} = \mu(x), \tag{17}$$

and equilibrium profit is $\pi^*(x) = \frac{\mu(x) h^*(x)}{D} = F_\Pi^{-1}(z(x))$.

Hence $\mu'(x) = m'(z(x))\, z'(x)$ and $h^{*\prime}(x) = D y'(z(x))\, z'(x)$. These two unknown functions satisfy condition (12), which implies $\frac{y'(z(x))z'(x)}{y(z(x))} = \frac{(1-m'(z(x))z'(x))}{m(z(x))}$, so that

$$z'(x) = \frac{y(z(x))}{(m(z(x))\, y(z(x)))'} = \frac{F_Y^{-1}(z(x))}{\left(F_\Pi^{-1}(z(x))\right)'}$$

or, inverting, we can write $x'(z) = \frac{dx}{dz} = \frac{(m(z)y(z))'}{y(z)} = \frac{\left(F_\Pi^{-1}(z)\right)'}{F_Y^{-1}(z)}$. Integrating,

$$x(z) = x(0) + \int_0^z \frac{\left(F_\Pi^{-1}(r)\right)'}{F_Y^{-1}(r)} dr = \underline{x} + \Psi(z). \tag{18}$$

Because $\Psi'(z) = \frac{\left(F_\Pi^{-1}(z)\right)'}{F_Y^{-1}(z)} > 0$, the required correspondence between z and x is $z = \Psi^{-1}(x - \underline{x})$. This makes clear that we can normalize $\underline{x}$ because the x values are all relative to this base (nonetheless, we retain $\underline{x}$ in what follows). The distribution of quality-cost is thus given by

$$F_X(x) = z(x) = \Psi^{-1}(x - \underline{x}), \tag{19}$$

thus we have an expression for $f_X(x)$. Now that we know $z(x)$, the remaining unknowns can now be backed out. ■

Example (ρ-linear demands and uniform quality-cost distribution)

Suppose that $F_Y(y) = \frac{(1+\rho)y^\rho - 1}{\rho}$, $y \in \left[\left(\frac{1}{1+\rho}\right)^{1/\rho}, 1\right]$, and $F_\Pi(\pi) = \frac{(1+\rho)\pi^{\rho/(1+\rho)} - 1}{\rho}$, $\pi \in \left[\left(\frac{1}{1+\rho}\right)^{(1+\rho)/\rho}, 1\right]$. Hence $F_Y^{-1}(z) = \left(\frac{\rho z+1}{1+\rho}\right)^{1/\rho}$ and $F_\Pi^{-1}(z) = \left(\frac{\rho z+1}{1+\rho}\right)^{(1+\rho)/\rho}$. The ratio of these two yields $m(z) = \frac{\rho z+1}{1+\rho}$. Because $\left(F_\Pi^{-1}(z)\right)' = \left(\frac{\rho z+1}{1+\rho}\right)^{1/\rho}$, we can write $x'(z) = \frac{\left(F_\Pi^{-1}(z)\right)'}{F_Y^{-1}(z)} = 1$, and hence, from (18) $x(z) = x(0) + \int_0^z dr = \underline{x} + \Psi(z)$, or $x(z) = \underline{x} + z$, i.e. $F_X(x) = z = x - \underline{x}$. Then $\mu(x) = \frac{\rho(x-\underline{x})+1}{1+\rho}$, and so $y(x) = F_Y^{-1}(z(x)) = \left(\frac{\rho(x-\underline{x})+1}{1+\rho}\right)^{1/\rho}$, and because $h^*(x) = Dy(x)$,

$h(.)$ is therefore a $\rho-$linear demand function. Note that $y(\underline{x}) = \left(\frac{1}{1+\rho}\right)^{1/\rho}$, as verified by the lower bound, $\underline{y}$, while the upper bound condition $\bar{y} = 1$ implies that $\bar{x} - \underline{x} = 1$.[20] Lastly, $\lim_{\rho\to 0} y(x) = \lim_{\rho\to 0}\left(\frac{\rho(x-\underline{x})+1}{1+\rho}\right)^{1/\rho} = \exp(x-\underline{x})$ gives the logit model with a unit mark-up.

5.3 Deriving costs and the quality-cost to cost relation

If we also have the price distribution, $F_P(p)$ then we can furthermore back out the cost distribution, $F_C(c)$, and the quality-cost to cost relation $\beta(c)$. The steps are as follows. First, determine the mark-up distribution, $F_M(m)$, from the mark-up relation $\mu(x)$ and the quality-cost distribution, $F_X(x)$. Then, use $F_M(m)$ with the price distribution to uncover the underlying cost distribution, $F_C(c)$. Matching this with $F_X(x)$ uncovers the relation between cost and quality-cost (the function $\beta(c)$ from Section 4).

In the sequel we shall consider the special case of strictly log-concave $h(.)$, which implies (and is implied by) $\mu'(x) \in (0,1)$. Knowledge of $F_Y(y)$ and $F_\Pi(\pi)$ determines $\mu(x)$ and $F_X(x)$ from Theorem 5. Then we can derive the mark-up distribution, $F_M(m)$, from the relation:

$$F_M(m) = \Pr(M < m) = \Pr(\mu(X) < m) = F_X\left(\mu^{-1}(m)\right), \tag{20}$$

where we have used that $\mu'(x) > 0$. Notice that there are implied properties on the resulting mark-up distribution function. Because $F_X(x) = F_M(\mu(x))$ then $\mu'(x) < 1$ implies $f_M(\mu(x)) > f_X(x)$.[21]

With the mark-up function $F_M(m)$ thus determined, suppose that $F_P(p)$ is known. To uncover $F_C(c)$ requires knowing whether costs and mark-ups move together or not. Suppose

[20] Using the definition of D from (8) we get $\mathcal{V}_0 = D\left(1 - M(1+\rho)^{-1-1/\rho}\right)$. Hence we can either normalize $\mathcal{V}_0 = 1$ here so $D = \frac{1}{\left(1-M(1+\rho)^{-1-1/\rho}\right)}$, or else we can normalize $h^*(\underline{x}) = Dy(\underline{x}) = 1$ so that $D = \left(\frac{1}{1+\rho}\right)^{-1/\rho}$ and thus $\mathcal{V}_0 = \left(\frac{1}{1+\rho}\right)^{-1/\rho}\left(1 - M(1+\rho)^{-1-1/\rho}\right)$. In either case, we are at liberty to set $\underline{x} = 0$.

[21] Note also that $\mu'(x) = \frac{f_X(x)}{f_M(\mu(x))} > 0$, as desired for a strictly log-concave h.

that it is known that they do, then higher costs also entail higher prices. In this case we can match distributions by choosing a common level z of the distributions and write

$$F_P(p) = F_C(c) = F_M(m) = z; \quad \text{with } p - c = m.$$

Then we can uncover $F_C(c)$ through the relation

$$F_C^{-1}(z) = F_P^{-1}(z) - F_M^{-1}(z). \tag{21}$$

With $F_C(c)$ thus determined, the final primitive to determine is the relation between quality-cost and cost. We can here proceed as we did in Section 4, and recover the relation between $F_X(x)$ and $F_C(c)$ via the transformation function $\beta(c)$ (see Figure 1). Given that $\beta(c)$ is an increasing function, we have (because $x = F_X^{-1}(z)$ by (15)):

$$x = \beta(c) = F_X^{-1}(F_C(c)). \tag{22}$$

The results above are summarized as follows.

Theorem 6 *Consider a demand model (7) under monopolistic competition, where it is known that $h(.)$ is strictly log-concave and quality-cost increases with cost. Assume that the distributions of profit and output and price are known. Then the quality-cost distribution is given by (19), the mark-up function is given by (17), and equilibrium quasi-demand is given by (16). The cost distribution is given by (21), and the quality-cost bridge is given by (22).*

Example

Suppose we know that $0 < a < b$ $(= a + 1)$ and $k > 0$, and $F_\Pi(\pi) = 2\sqrt{k\pi} - a$ $(= z)$ for $\pi \in \left[\frac{a^2}{4k}, \frac{b^2}{4k}\right]$, $F_Y(y) = 2ky - a$ $(= z)$ for $y \in \left[\frac{a}{2k}, \frac{b}{2k}\right]$, and $F_P(p) = k_p\left(p - \underline{p}\right)$ with $k_p < 2$.

Then $F_{\Pi}^{-1}(z) = \frac{1}{k}\left(\frac{z+a}{2}\right)^2$ and $F_Y^{-1}(z) = \frac{z+a}{2k}$; the relation between z and x is given by (18) as

$$\int_0^z \frac{\left(F_{\Pi}^{-1}(r)\right)'}{F_Y^{-1}(r)} dr = \int_0^z dr = z = \Psi(z) = x - \underline{x},$$

so $z = \Psi^{-1}(x - \underline{x}) = F_X(x) = x - \underline{x}$, and $F_X(x)$ is uniform on $[\underline{x}, \underline{x}+1]$. Then the mark-up is

$$\mu(x) = \frac{F_{\Pi}^{-1}(x - \underline{x})}{F_Y^{-1}(x - \underline{x})} = \frac{x - \underline{x} + a}{2}.$$

Then $h^*(x) = D\frac{x-\underline{x}+a}{2k}$ (using (16)), and the associated h function is $h(v-p) = h(x-m) = \frac{D}{k}(x - \underline{x} + a - m)$ (from Theorem 4) so this is a linear quasi-demand. Using the specification (8) gives $D = \mathcal{V}_0 / \left(1 - \frac{M}{4k}(1+2a)\right)$.[22]

Because $F_M(m) = F_X(\mu^{-1}(x))$, and $\mu(x) = \frac{x-\underline{x}+a}{2}$, then $F_M(m) = 2m - a$ (for $m \in \left[\frac{a}{2}, \frac{b}{2}\right]$).

We can now find the cost distribution from the price distribution, using (21): $F_C^{-1}(z) = F_P^{-1}(z) - F_M^{-1}(z)$. This gives $F_C^{-1}(z) = \left(\frac{z}{k_p} + \underline{p}\right) - \left(\frac{z+a}{2}\right) = z\left(\frac{1}{k_p} - \frac{1}{2}\right) + \underline{c}$. Hence we have a uniform distribution, $F_C(c) = \left(\frac{2k_p}{2-k_p}\right)(c - \underline{c})$ (where $\underline{c} = \underline{p} - \underline{m}$). We can now determine the bridge function $\beta(c)$ from (22): $x = \beta(c) = F_X^{-1}(F_C(c))$. That is, $\beta(c) = \left(\frac{2k_p}{2-k_p}\right)(c - \underline{c}) + \underline{x}$ (equivalently, $x - \underline{x} = \left(\frac{2k_p}{2-k_p}\right)(c - \underline{c})$); here $\beta'(c) > 0$, as stipulated, given the restriction $k_p < 2$.[23]

An analogous analysis to that in Theorem 6 applies for $\mu'(x) < 0$, which corresponds to strictly log-convex demand (although recall we still require demand to be (-1)-concave to guarantee a unique maximum to firms' profit functions). In the log-convex case, mark-ups decrease with firm quality-cost and the relation we uncover below between the mark-up distribution and the quality-cost one is inverted, analogously to the inversion of the $\beta(c)$ function analyzed in Section 4.2. Finally, if $\mu(x)$ is non-monotone, the primitives cannot be backed out on their full support, analogous to the discussion in Section 4.2. Note that $\mu'(x) = 0$

[22] As discussed above, we can normalize $\mathcal{V}_0 = 1$, or else $h^*(\underline{x}) = D\frac{a}{2k} = 1$ so $D = \frac{2k}{a}$. We can also set $\underline{x} = 0$.
[23] Such a linear bridge function can arise from endogenous quality-cost choices of heterogeneous firms: see the example in Section 4.1.

corresponds to the Logit case, which we have already analyzed in full.

6 Long run Logit

Here we develop the long-run analysis of the logit model following recent directions in Trade models, and emphasize the shape of the equilibrium distributions that ensue. A fuller (more general) analysis along the lines of the previous section would follow similar lines, but here we aim for simplicity. We assume in the groove of Melitz (2003) that firms first pay a cost K_1 to get a quality-cost draw, then they pay K_2 to actively produce. We solve backwards.[24] To put in play market size effects, we introduce market size (number of consumers) N (which was normalized to 1 in the analysis so far).

For a given mass, M, of firms that have paid K_1, equilibrium involves all sufficiently good types paying the subsequent fixed cost K_2. The firm of type $\tilde{x}$ just covers its cost, K_2, and $\tilde{x} \geq \underline{x}$ if K_2 is low enough. All types $x \geq \tilde{x}$ will produce (because profits increase in x by Proposition 1). The gross profit of firm with quality-cost x is now

$$\pi\left(x, \tilde{x}\right) = \mu N \frac{\exp\left(\frac{x}{\mu}\right)}{D\left(M, \tilde{x}\right)},$$

where $D\left(M, \tilde{x}\right) \equiv M \int_{u \geq \tilde{x}} \exp \frac{u}{\mu} f_X\left(u\right) du + \mathcal{V}_0$, $\underline{x} \leq \tilde{x}$ (see (5)). $D\left(M, \tilde{x}\right)$ is decreasing in $\tilde{x}$ so that the profit of the marginal firm, $\pi\left(\hat{x}, \hat{x}\right)$ is increasing in $\hat{x}$. Hence, as long as $\pi\left(\underline{x}, \underline{x}\right) < K_2$, there is a unique cut-off value $\hat{x}$ such that

$$\pi\left(\hat{x}, \hat{x}\right) = K_2. \tag{23}$$

This is the case we consider: otherwise, all firms enter, and all make strictly positive profits.

Once a firm has paid the cost K_1 to get a draw, it has a probability $1 - F_X\left(\hat{x}\right)$ to get a good enough draw, and to be active. The mass of potentially active firms, M, is determined at

[24] If X is observed by firms before entry, only the first part of the analysis should be retained.

the first step via the zero-profit condition:

$$\mu N \frac{\int_{u\geq\hat{x}} \exp\left(\frac{u}{\mu}\right) f_X(u)\, du}{D(M,\hat{x})} - K_2 \int_{u\geq\hat{x}} f_X(u)\, du = K_1. \tag{24}$$

The first term is the expected gross profit of a firm which has paid the entry cost K_1 to get a draw. The second is the fixed (continuation) cost, to be paid by all firms with a draw of at least $\hat{x}$. Inserting (23) in (24) gives

$$\int_{u\geq\hat{x}} \left(\exp\left(\frac{u-\hat{x}}{\mu}\right) - 1\right) f_X(u)\, du = \frac{K_1}{K_2}. \tag{25}$$

The LHS is monotonically decreasing in $\hat{x}$ so there is a *unique* solution for $\hat{x}$ that depends only on the parameters in (25) – it is independent of market size N and of $\mathcal{V}_0$ (Bertoletti and Etro, 2014, show a similar neutrality result). The condition for an interior solution is that

$$\int_{u\geq\underline{x}} \left(\exp\left(\frac{u-\underline{x}}{\mu}\right) - 1\right) f_X(u)\, du < \frac{K_1}{K_2}. \tag{26}$$

Otherwise, all firms are in the market. Given the solution for $\hat{x}$, we can then determine M from (23) with $D(M,\hat{x})$ and defining $x^c = \max\{\hat{x}, \underline{x}\}$:

$$M = \frac{\frac{\mu N}{K_2}\exp\left(\frac{x^c}{\mu}\right) - \mathcal{V}_0}{\int_{u\geq x^c} \exp\left(\frac{u}{\mu}\right) f_X(u)\, du}. \tag{27}$$

Therefore $M > 0$ if

$$Z^c \equiv \frac{\mu N}{K_2} \exp\left(\frac{x^c}{\mu}\right) > \mathcal{V}_0. \tag{28}$$

Otherwise, there is no entry ($M = 0$). Note that condition (28) depends on both K_1 and K_2 since $\hat{x}$ depends on K_1/K_2.

Proposition 6 *(Logit, long-run) Consider the Logit Monopolistic Competition model, with cost K_1 to get a quality-cost draw, and cost K_2 to actively produce. Some firms enter if (28) holds; some of these entrants do not produce if (26) holds. Then the solution satisfies (23) and (27).*

This solution parallels that for the Melitz (2003) model.[25] Some comparative static properties readily follow. The elasticity of M with respect to N is (using (28)) $\left(1 - \frac{\mathcal{V}_0}{Z^c}\right)^{-1} > 1$. Hence, if the market is covered ($\mathcal{V}_0 = 0$), the number of firms is proportional to market size. Otherwise, firm numbers *more than double* (see also Melitz, 2003; Bertoletti and Etro, 2014, show the opposite case can arise when there are income effects).

Thus, the long-run uniquely determines $\hat{x}$ and M. From these, the long-run distribution of quality-cost is $\tilde{F}_X(x) = \frac{F_X(x) - F_X(\hat{x})}{1 - F_X(\hat{x})}$ for $x \in [\hat{x}, \bar{x}]$, and then Theorems 1 and 2 hold. Moreover, the inheritance properties of the key distributions still apply. In particular, if F_X is an exponential distribution for F_X then so is $\tilde{F}_X$ ($\tilde{F}_X = 1 - \exp(-\lambda(x - \hat{x}))$). Thus profits and outputs are Pareto, and then we can link to the cost and price distributions as we did before: *the size distribution of output and profit is Pareto, with shape parameter $\lambda\mu$.*

The comparative statics results of Proposition 3 are readily amended: for fixed $\underline{x}$ at the bottom of the support, a higher $\mathcal{V}_0$ decreases profits, while a higher μ raises them for low quality-cost firms and reduces them for high quality-cost firms. Now, when the lower bound is endogenous, it is clear from (25) that $\hat{x}$ is unchanged when $\mathcal{V}_0$ rises, but M falls (from (27)). Thus the effect is just as before, except now milder by the exit of firms. For higher μ, by (27) $\hat{x}$ falls,[26] so that increased taste heterogeneity increases the range of firm types that will stay in the market after their initial draw. However, the number of firms taking the first draw (M) may increase or decrease – high quality firm types get lower profits (cf. Proposition 3), and

[25] We can readily include a second threshold for exporting firms in an international trade context.

[26] For example, for the exponential distribution of quality-cost, $\hat{x} = \underline{x} + \frac{1}{\lambda} \ln\left(\frac{K_2}{K_1}\frac{1}{\lambda\mu - 1}\right)$, which is decreasing in μ. The corresponding mass of entering firms is $M = \frac{N}{\lambda K_1} - \frac{\mathcal{V}_0}{\lambda\mu}\frac{K_2}{K_1}\exp\left(\frac{-\hat{x}}{\mu}\right)$.

this may decrease the desire to enter.

7 CES models

A flurry of recent contributions use the CES and variants thereof (e.g., Dhingra and Morrow, 2013, Zhelobodko, Kokovin, Parenti, and Thisse, 2012, Bertoletti and Etro, 2014, etc.). Most noticeably, it has enjoyed a huge spurt in popularity in the new international trade literature.[27]

Here we apply the distributional analysis to the CES. We start with the standard CES monopolistic competition model with heterogeneity only in firms' unit production costs (this is the basic Melitz, 2003, approach). Hence, all economic distributions (prices, output, profit, and revenue) are tied down by the cost distribution.

A central distribution in the literature has been the Pareto. We show that all relevant distributions are Pareto if any one is (caveat: for prices and costs it is the distribution of the reciprocal that is Pareto). This result we term the *Pareto circle*. To put this another way, if we posit that the reciprocal of costs is Pareto distributed (equivalently, costs have a power distribution), then so is the reciprocal of prices, and the other variables (output, revenue, and profit) are all Pareto distributed. It is not possible to have (for example) a Pareto distribution for profits and (another) Pareto distribution for prices in the CES model. The Pareto circle cannot be escaped if one element is Pareto. Similar results hold for other distributions, yielding a more general *CES circle*.

Following Baldwin and Harrigan (2011) and Feenstra and Romalis (2014), we therefore introduce a further dimension of heterogeneity, just as we did for the logit, and again interpreted as "quality." As with the logit analysis we link the two distributions via a bridge function (analogous to $\beta(c)$ above) that writes quality as a function of cost. Doing this then enables us to get two linked groups of distributions. In one group are profit and revenue, and in the

[27] Although note that Fajgelbaum, Grossman, and Helpman (2009) take a nested multinomial logit approach.

other are costs and prices, while output forms a convex combination. Our leading example is a bridge function that delivers Pareto distributions in each group. We first develop the analysis for cost heterogeneity alone.

7.1 Standard CES model

Several forms of CES representative consumer utility functions are prevalent in the literature. We nest these into one embracing form. The CES representative consumer involves a sub-utility functional for the differentiated product $\chi = \left(\int_{\Omega} q(\omega)^{\rho}\, d\omega\right)^{1/\rho}$ with $\rho \in (0,1)$ (with $\rho = 1$ being perfect substitutes, and $\rho \to 0$ being independent demands), and the q's are quantities consumed of the differentiated variants. Common forms of representative consumer formulation are (i) Melitz model (see also Dinghra and Morrow, 2013) where $U = \chi$ so there is only one sector); (ii) the classic Dixit-Stiglitz (1977) case much used in earlier trade theory, $U = \chi q_0^{\eta}$ with $\eta > 0$, where q_0 is consumption in an outside sector; (iii) $U = \ln \chi + q_0$, which constitutes a partial equilibrium approach in the sense that there are no income effects (see Anderson and de Palma, 2000). The first two involve unit income elasticities, hence their popularity in Trade models. Utility is maximized under the budget constraint $\int_{\Omega} q(\omega)\, p(\omega)\, d\omega + q_0 \leq I$, where I is income.

The next results are quite standard. For a given set of prices and a set Ω of active firms, Firm i's demand (output) is:

$$q_i = \frac{\Xi(I)}{p_i} \frac{p_i^{\frac{\rho}{\rho-1}}}{\int_{\omega\in\Omega} p(\omega)^{\frac{\rho}{\rho-1}}\, d\omega}, \tag{29}$$

where $\Xi(I)$ is I for case (i), $\frac{I}{1+\eta}$ for case (ii) (which clearly nests case (i) for $\eta = 0$); and 1 for the last case. In each case, $\Xi(I)$ is the amount spent on the differentiated commodity in aggregate. In the sequel we follow case (iii); the others are similarly straightforward.

The price solves $\max\limits_{p_i} \frac{(p_i - c_i)}{p_i} p_i^{\frac{\rho}{\rho-1}}$, so $p_i = \frac{c_i}{\rho}$, and the Lerner index is $\frac{p_i - c_i}{p_i} = (1-\rho)$. Given such pricing, Firm i's equilibrium output is

$$q_i = \rho \Xi(I) \frac{c_i^{\frac{\rho}{\rho-1}-1}}{D_C}, \tag{30}$$

where $D_C = M \int c(u)^{\frac{\rho}{\rho-1}} f_C(u)\, du$, and $f_C(.)$ is the unit cost density. Firm i's equilibrium profit is a constant fraction of its sales revenue, $r_i = p_i q_i$, with $r_i = \Xi(I) \frac{c_i^{\frac{\rho}{\rho-1}}}{D_C}$, so $\pi_i = (1-\rho) r_i$.

We can now tie together the various equilibrium distributions with the help of the following straightforward result, which tells us how distributions are modified by powers and multiplicative transformations. These transformations relate profit, revenue, output, price reciprocal $(1/p)$, cost reciprocal $(1/c)$ in the CES model.

Lemma 2 *(Transformation) Let $F_X(x)$ be the CDF associated to a random variable X. Then the CDF of kX^θ with $k > 0$ is $F_{kX^\theta}(x) = F_X\left[\left(\frac{x}{k}\right)^{\frac{1}{\theta}}\right]$ for $\theta > 0$, and $F_{kX^\theta}(x) = 1 - F_X\left[\left(\frac{x}{k}\right)^{\frac{1}{\theta}}\right]$ for $\theta < 0$.*

For example, power distributions beget power distributions under positive power transforms and Pareto distributions under negative power transforms. Furthermore, normal distributions beget normal distributions in both cases, due to the symmetry of the normal distribution, etc. We refer to pairs of distributions with the same functional forms but different parameters as being in the same *class* (e.g., Pareto, power, normal distributions are all classes).

Proposition 7 *(CES circle) For the CES, the distributions of profit, revenue, output, price reciprocal and cost reciprocal are all in the same class.*

Proof. From the analysis above, all these variables for the CES involve positive power transformation and/or multiplication by positive constants. Profit is proportional to revenue; price is proportional to cost, and likewise for their reciprocals. From (30), equilibrium output, q_i, is related to the cost reciprocal, $1/c_i$, by a positive power and a positive factor. The other relations follow directly. ■

In particular, if any one of these distributions is Pareto (resp. power), then they all are Pareto (resp. power) class, although they have different parameters. Similarly, if one is normal (resp. log-normal) then all are normal (resp. log-normal). This result we term the *CES-circle*. It means that the standard CES model with cost heterogeneity alone cannot deliver (say) Pareto distributions for *both* profit and prices. Indeed, if profit is Pareto distributed, then price must follow a power distribution. We next introduce quality heterogeneity to break the CES-circle.

7.2 CES quality-enhanced model

To now extend the model to allow for quality differences across products, we rewrite the sub-utility functional as $\chi = \left(\int_\Omega z(\omega)^\rho d\omega\right)^{1/\rho}$ with $\rho \in (0,1)$ and interpreting $z = vq$ as the quality-adjusted consumption (see Baldwin and Harrigan, 2011, and Feenstra and Romalis, 2014). The corresponding demands are:

$$q_i = \frac{\Xi(I)}{p_i} \frac{\hat{p}_i^{\frac{\rho}{\rho-1}}}{\int_{\omega\in\Omega} \hat{p}(\omega)^{\frac{\rho}{\rho-1}} d\omega}, \tag{31}$$

where we have defined $\hat{p}_i = p_i/v_i$ which is interpreted as the price per unit of "quality" and $\Xi(I)$ is as above for the three different cases (the amount spent on the differentiated commodity). The key feature of (31) is that p_i enters both with and without quality in the denominator. The standard model (29) ensues when all the v's are the same.

Under monopolistic competition, Firm i's equilibrium price solves $\max_{p_i} \frac{(p_i - c_i)}{p_i} \hat{p}_i^{\frac{\rho}{\rho-1}}$ so the pricing solution $p_i = \frac{c_i}{\rho}$ still holds. Hence, using $x_i = v_i/c_i$,[28] which we refer to as quality/cost, the equilibrium profit is[29]

$$\pi_i = (1-\rho)\,\Xi(I) \frac{x_i^{\frac{\rho}{1-\rho}}}{\int_{\omega\in\Omega} x(\omega)^{\frac{\rho}{1-\rho}} d\omega} = (1-\rho)\, r_i. \tag{32}$$

Equilibrium profit is still a fraction $(1-\rho)$ of revenue. This implies that profit, sales revenue,

[28] This is quality/cost whereas the logit has quality-cost.

[29] Proposition 1 holds here too except for the CES proportional mark-up.

and quality/costs distributions are in the same class.[30]

Price and cost distributions are still in the same class, but reciprocal costs and profits are not necessarily so. How the cost and profit distributions are linked is determined by the relation between cost and quality. A functional relation between cost and quality/cost ties down the bridging relation, and the distributions on the "other" side.

A central example of a bridging function is $x = c^{\gamma}$ so that quality/cost is increasing with cost (so quality rises faster than cost) if $\gamma > 0$ and it is decreasing if $\gamma < 0$. The latter case is embodied in the standard CES model above where $\gamma = -1$ and so "better" firms are those with lower costs. The former case effectively corresponds to Feenstra and Romalis (2014). The advantage of the constant elasticity bridging function is that it allows us to deploy results (Lemma 2) on applying power transforms to random variables.

Because profits are proportional to $x_i^{\frac{\rho}{1-\rho}}$ (see (32)), they are proportional to $c_i^{\gamma\frac{\rho}{1-\rho}}$. Hence if $\gamma > 0$ profits are in the same distribution class as costs. So then too are revenues and quality-costs. But if $\gamma < 0$, profits, revenues and quality-costs are in the "opposite" (or "inverse") class - this is the generalization of the earlier standard CES result. Prices, of course, are in the same class as costs, but output is more intricate because it draws its influences from both sides. Indeed, output is proportional to $x_i^{\frac{\rho}{1-\rho}}/c_i$ (see (31)) which equals $c_i^{\gamma\frac{\rho}{1-\rho}-1}$ under the constant elasticity formulation. This implies that for $\gamma < 1 - \frac{1}{\rho} < 0$ the output distribution is in the inverse class, while otherwise it is in the same class.

For what follows, we define two distributions as in the same class if they have the same functional form. One distribution is the *inverse* of another one if it is the survival function of the other distribution. A summarizing statement:

[30] Profits are increasing in x so that firms would like this as large as possible. As we did with Logit, we can link cost and quality through a type of "production" function and have (heterogeneous) firms choose their x. Along the same lines as Feenstra and Romalis (2014), we can let $v = l^{\alpha}$ be the quality produced at cost $wl + \phi$ with ϕ a firm-specific productivity shock. Then, maximizing $x = l^{\alpha}/(wl + \phi)$ gives the optimized value relation between cost and quality as $x = \left(\frac{\alpha}{w}\right)^{\alpha} c^{\alpha-1}$ and so the bridge function takes a power form. Here it is decreasing (and depends on the fundamental via $x = \phi^{\alpha-1}$).

Proposition 8 *(Breaking the CES circle) Consider the quality-enhanced CES model of monopolistic competition with $x = c^\gamma$. Then:*

i) the equilibrium price distribution mirrors the unit cost distribution;

ii) equilibrium profits, sales revenue, and quality/cost are in the same distribution class for $\gamma < 0$ and in the inverse class for $\gamma > 0$;

iii) equilibrium output is in the inverse distribution class for $\gamma < 1 - \frac{1}{\rho} < 0$, and in the same distribution class for $\gamma > 1 - \frac{1}{\rho}$.

Note that inverse distributions take the same form for symmetric distributions such as the Normal, so then all distributions belong to the same class – once a normal, always a normal.

Take the example of a Pareto distribution for costs. First, prices are also Pareto distributed. Second, profits, revenue, and quality/cost are Pareto distributed for $\gamma > 0$ and power distributed for $\gamma < 0$ (they are independent of cost if $\gamma = 0$). Third, output is power distributed for $\gamma < 1 - \frac{1}{\rho} < 0$, and Pareto distributed for $\gamma > 1 - \frac{1}{\rho}$. If costs are power distributed, Pareto and power are reversed in the above statements. Hence, we resolve the puzzle of getting Pareto distributions for both prices and profits by including the appropriate bridge function.

Proposition 8(ii) indicates that quality/cost and profits fall in the same distribution. For example, suppose that the distribution of quality/costs is Pareto: $F_X(x) = 1 - \left(\frac{\underline{x}}{x}\right)^\lambda$ and assume that $\lambda\frac{1-\rho}{\rho} > 1$. Then the size distribution of profit is Pareto with tail parameter $\alpha_\Pi = \lambda\frac{1-\rho}{\rho}$. The well-known claimed empirical regularity "80-20" rule (that the top 20% of firms account for 80% of sales) corresponds to a value α_Π of 1.161. The result here is that the profit tail parameter is the confluence of a preference parameter and a quality/cost distribution one.[31]

[31] Although why they yield the same constant across settings remains intriguing.

8 Conclusions

The basic ideas here are simple. Market performance depends on the economic fundamentals of tastes and technologies, and how these interact in the market-place.[32] The fundamental distribution of tastes and technologies feeds through the economic process to generate the endogenous distribution of economic variables, such as prices, outputs, and profits. By invoking the monopolistic competition assumption we get a straight feed-through from fundamental distributions to performance distributions.

Quality and cost differences are especially interesting for empirical work and studying asymmetric firms. A normal quality-cost distribution leads to a log-normal distribution of firm size, and an exponential quality-cost distribution generates a Pareto distribution. In the CES formulation, the assumed distribution of costs is also the equilibrium distribution of outputs: Pareto begets Pareto. This cycle is broken by allowing for quality heterogeneity in the CES.

The CES model has been the workhorse model of monopolistic competition with asymmetric firms. The Logit model gives some similar properties, while differing on others. For example, the simple CES has constant percentage mark-ups while the Logit has constant absolute mark-ups.[33] The Logit can be deployed for similar purposes as the CES, and has an established pedigree in its micro-economic underpinnings. It has a strong econometric backdrop which is at the heart of much of the structural empirical industrial organization revolution.

Our main results in the heart of the paper show how to back out the demand form from profit and output distributions, and hence, with the addition of knowledge of the price distribution, to recover all the primitives of the model. To do this, though, we relied on there being a one-dimensional underlying relationship between costs and quality-costs (which we also showed how to recover). One direction for future research is to consider a multi-dimensional relationship.

[32] Firm size distributions have recently come to the fore in Chris Anderson's (2006) work on the Long Tail of internet sales.

[33] The latter property is perhaps quite descriptive for cinema movies, DVDs, and CDs.

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Appendix 1

Proof of Proposition 2

i) We treat the case of output; profit is analogous. Mean output is:

$$y_{av} = \int_{w \geq \underline{y}} w f_Y(w)\, dw = \frac{M\xi}{M\xi + \mathcal{V}_0},$$

where $\xi = \int_{u \geq \underline{x}} \exp\left(\frac{u}{\mu}\right) f_X(u)\, du = \exp\left(\frac{\bar{x}}{\mu}\right) - \frac{1}{\mu}\int_{\underline{x}}^{\bar{x}} \exp\left(\frac{u}{\mu}\right) F_X(u)\, du$, so that a fosd decrease in $F_X(u)$ (holding $\bar{x}$ constant) raises ξ (strictly if and only if $\mathcal{V}_0 > 0$) and hence raises y_{av}.

ii) Given that $\exp\left(\frac{u}{\mu}\right)$ is convex and increasing, ξ increases (strictly if and only if $\mathcal{V}_0 > 0$) with a mean-preserving spread in $f_X(.)$. The result follows immediately.

Proof of Proposition 3

By Theorem 1, $F_Y(y) = F_X(\mu \ln(yD))$. Because D is increasing in $\mathcal{V}_0$, then $F_Y(y)$ is increasing in $\mathcal{V}_0$, so that output is first-order stochastically dominated when $\mathcal{V}_0$ rises. The argument for profits is analogous because $F_\Pi(\pi) = F_\Pi\left(\mu \ln\left(\frac{\pi D}{\mu}\right)\right)$ by Theorem 1.

The effects on $F_Y(y)$ of an increase in μ are determined by the behavior of $\mu \ln(yD)$ as a function of μ. The derivative is $\ln y + \frac{d(\mu \ln D)}{d\mu}$; the second term is independent of y, and we show in Lemma 1 that it is positive. Note that $y < 1$ (because the sum of outputs is below 1), so that $\ln y < 0$. Therefore $F_Y(y)$ goes down with μ for y low enough (i.e., for quality-costs low enough: see Proposition 1), and goes up for y high enough (i.e., for quality-costs high enough). The effects on $F_\Pi(\pi)$ of higher μ are determined by when $\mu \ln\left(\frac{\pi D}{\mu}\right)$ is increasing or decreasing in μ. The derivative is $\ln\frac{\pi}{\mu} + \frac{d(\mu \ln D)}{d\mu} - 1 = \left[\ln y + \frac{d(\mu \ln D)}{d\mu}\right] - 1$, and so for intermediate output (or profit) levels, a higher μ may increase $F_Y(y)$, but decrease $F_\Pi(\pi)$.

Proof of Proposition 4

We first calculate the logit denominator, D, from (5), using the exponential CDF

$$F_X(x) = 1 - \exp(-\lambda(x - \underline{x})), \quad \lambda > 0,\ \underline{x} > 0, \quad x \in [\underline{x}, \infty), \tag{33}$$

with density $f_X(x) = \lambda \exp(-\lambda(x - \underline{x}))$. Assume $\lambda\mu > 1$ for D to be bounded. Integrating,

$$D = \frac{M\lambda\mu}{\lambda\mu - 1} \exp\left(\frac{\underline{x}}{\mu}\right) + \mathcal{V}_0, \tag{34}$$

which is positive for any $\mathcal{V}_0$, since $\lambda\mu > 1$. Now, from Theorem 1,

$$F_\Pi(\pi) = 1 - \exp\left(-\lambda\left(\mu \ln\left(\frac{\pi D}{\mu}\right) - \underline{x}\right)\right) = 1 - \left(\frac{\pi D}{\mu}\right)^{-\lambda\mu} \exp(\lambda\underline{x}).$$

$\underline{\pi}$ is the profit of the lowest quality-cost firm and solves $F_\Pi(\underline{\pi}) = 0$, and thus verifies the expected property $\underline{\pi} = \frac{\mu}{D} \exp\left(\frac{\underline{x}}{\mu}\right)$. Inserting this value back into $F_\Pi(\pi)$ gives the expression in Proposition 4. The output distribution follows from the profit distribution:

$$F_Y(y) = \Pr(Y < y) = \Pr\left(\frac{\Pi}{\mu} < y\right) = F_\Pi(\mu y) = 1 - \left(\frac{\underline{\pi}}{\mu y}\right)^{\lambda\mu} = 1 - \left(\frac{\underline{y}}{y}\right)^{\lambda\mu},$$

where the lowest output, $\underline{y}$, is associated to the lowest profit, $\underline{\pi} = \mu\underline{y}$.

The last statement follows from Theorem 1: starting with a Pareto distribution for output or profit implies an underlying exponential distribution for quality-cost. The lowest quality-cost is given by the condition $\underline{y} = \frac{\mu}{D} \exp\left(\frac{\underline{x}}{\mu}\right)$, so

$$\underline{\pi} = \mu\underline{y} = \frac{\mu}{\frac{M\lambda\mu}{\lambda\mu-1} \exp\left(\frac{\underline{x}}{\mu}\right) + \mathcal{V}_0} \exp\left(\frac{\underline{x}}{\mu}\right) < \frac{1}{M}\left(\mu - \frac{1}{\lambda}\right). \tag{35}$$

The condition $\lambda\mu > 1$ ensures the logit denominator D exists. Inverting (35) gives $\underline{x}$.

Proof of Theorem 2

Because $F_X(x)$ is continuous and increasing on support $[\underline{x}, \bar{x}]$ and because $\beta^{-1}(x)$ is continuous and increasing on support $[\underline{x}, \bar{x}]$ with $\beta^{-1}(\underline{x}) > 0$. Then $F_C(c) = \Pr(C < c)$ is uniquely defined and continuous and increasing on support $[\underline{c}, \bar{c}]$:

$$F_C(c) = \Pr\left(\beta^{-1}(X) < c\right) = \Pr(X < \beta(c)) = F_X(\beta(c)).$$

The last term is a continuous and increasing function of a continuous and increasing function, so $F_C(c)$ is recovered. Constructing $F_X(x)$ from $F_C(c)$ and $\beta(c)$ is completely analogous.

We now show how to construct a unique increasing $\beta(c)$ from the two distributions: let $F_X(x) = \Pr(X < x)$ and we postulate that there exists a continuous increasing function $\beta(C) = X$ and so $F_X(x) = \Pr(\beta(C) < x) = \Pr\left(C < \beta^{-1}(x)\right)$ which is then equal to $F_C\left(\beta^{-1}(x)\right)$. Now, since $F_X(x) = F_C\left(\beta^{-1}(x)\right)$, then $\beta^{-1}(x) = F_C^{-1}(F_X(x))$, so $\beta(x) = \left[F_C^{-1}(F_X(x))\right]^{-1}$ and $\beta(x) = F_X^{-1}(F_C(x))$. This is clearly increasing and continuous in x as desired.

The claim in the Theorem is shown because $F_X(x)$ can be used to construct the other distributions on its leg, and can be constructed from them; and likewise for $F_C(c)$.

Proof of Lemma 1

For the discrete version of the Logit model,

$$y_j = \frac{\exp\left(\frac{r_j}{\mu}\right)}{\sum_{l=0...n} \exp\left(\frac{r_l}{\mu}\right)} < 1, \qquad j = 1..n$$

where r_l is an arbitrary scale value independent of μ. Taking logs and rearranging,

$$\ln\left(\sum_{l=0...n} \exp\left(\frac{r_l}{\mu}\right)\right) = \frac{r_j}{\mu} - \ln y_j,$$

where the LHS is the "log-sum" formula for the indirect utility, V, for the logit model. Summing over j and using for weights the choice probabilities, y_j (which sum to one):

$$V = \mu \ln\left(\sum_{j=0...n} \exp\left(\frac{r_j}{\mu}\right)\right) = \sum_{j=0...n} r_j y_j - \mu \sum_{j=0...n} y_i \ln y_i. \tag{36}$$

This expression shows the indirect utility function V is equal to the average deterministic utility plus the Shannon measure of information (which is positive, since $y_i < 1$). Thus, the Shannon (1948) statistic provides a measure of the aggregate benefit from variety. Note that the same expression can be obtained by comparing the direct and the indirect utility functions for the representative consumer associated to the Logit model (see its formulation in Chapter 2 of Anderson et al. 1992). The derivative of V is :

$$\begin{aligned}\frac{dV}{d\mu} &= \ln\left(\sum_{j=0\ldots n}\exp\left(\frac{r_j}{\mu}\right)\right) - \frac{1}{\mu}\sum_{j=0\ldots n}\frac{\exp\left(\frac{r_j}{\mu}\right)}{\sum_{j=0\ldots n}\exp\left(\frac{r_j}{\mu}\right)}r_i \\ &= \ln\left(\sum_{j=0\ldots n}\exp\left(\frac{r_j}{\mu}\right)\right) - \frac{1}{\mu}\sum_{i=0\ldots n} r_i y_i.\end{aligned}$$

Now, using (36) above, the derivative of the indirect utility function is equal (up to a multiplicative factor) to the Shannon measure of information (or entropy), which is positive:

$$\frac{dV}{d\mu} = -\sum_{i=0\ldots n} y_i \ln y_i > 0.$$

Appendix 2 Distribution details (NOT FOR PUBLICATION)

Proof of Theorem 1

We first seek the distribution of outputs, $F_Y(y) = \Pr(Y < y)$, that is generated from the primitive distribution of quality-cost. First note from (4) that $Y = \frac{\exp\left(\frac{X}{\mu}\right)}{D}$, so:

$$F_Y(y) = \Pr\left(\frac{\exp\left(\frac{X}{\mu}\right)}{D} < y\right) = F_X(\mu \ln(yD)),$$

where D is given by (5). Because equilibrium profit is proportional to output ($\pi = \mu y$), we have a similar relation for the distribution of profit, $F_\Pi(\pi) = \Pr(\Pi < \pi)$:

$$F_\Pi(\pi) = \Pr\left(\mu\frac{\exp\left(\frac{X}{\mu}\right)}{D} < \pi\right) = F_X\left(\mu \ln\left(\frac{\pi D}{\mu}\right)\right),$$

where D is given by (5).

We next prove the converse result. We first determine the distribution of quality-costs consistent with a given observed distribution of output. Suppose that output has a distribution $F_Y(y)$. Applying the increasing transformation $y = \frac{1}{D}\exp\left(\frac{x}{\mu}\right)$, and $Y = \frac{1}{D}\exp\left(\frac{X}{\mu}\right)$, we get:

$$\begin{aligned} F_X(x) &= \Pr\left(\frac{1}{D}\exp\left(\frac{X}{\mu}\right) < \frac{1}{D}\exp\left(\frac{x}{\mu}\right)\right) \\ &= \Pr\left(Y < \frac{1}{D}\exp\left(\frac{x}{\mu}\right)\right) = F_Y\left(\frac{1}{D}\exp\left(\frac{x}{\mu}\right)\right). \end{aligned}$$

However, D is written in terms of $f_X(x)$, and we want to find the distribution solely in terms of $F_Y(y)$: this means writing D in terms of $f_Y(y)$. The corresponding expression, denoted D_y is derived below as (37). Similar reasoning gives the profit expression:

$$F_X(x) = \Pr(X < x) = \Pr\left(\Pi < \frac{\mu}{D}\exp\left(\frac{x}{\mu}\right)\right) = F_\Pi\left(\frac{\mu}{D}\exp\left(\frac{x}{\mu}\right)\right),$$

where the expression for D in terms of $f_\Pi(\pi)$ (i.e., D_π) is given in the Theorem and derived below as (38).

We now show here how to write the function D as a function of $f_Y(.)$ or $f_\pi(.)$.

We first find the value of D in terms of the distribution of Y. Recall (5):

$$D = M \int_{u \geq \underline{x}} \exp\left(\frac{u}{\mu}\right) f_X(u)\, du + \mathcal{V}_0.$$

Now, $\Pr(x < X) = \Pr(y < Y)$, so $y(x) = \frac{\exp\left(\frac{x}{\mu}\right)}{M \int \exp\left(\frac{u}{\mu}\right) f_X(u) du + \mathcal{V}_0}$; hence $D \int y(u) f_Y(u)\, du = D - \mathcal{V}_0$, and thus:

$$D_y = \frac{\mathcal{V}_0}{1 - M y_{av}}. \tag{37}$$

The denominator in the last expression is necessarily positive because $M y_{av}$ is total output, which is less than one when $\mathcal{V}_0 > 0$ because the market is not fully covered. Similarly, $F_X(x) = F_\Pi\left(\frac{\mu}{D} \exp\left(\frac{x}{\mu}\right)\right)$, so that

$$D_\pi = \frac{\mathcal{V}_0}{1 - \frac{M}{\mu}\pi_{av}}, \tag{38}$$

which is now expressed as a function of $f_\Pi(.)$, and where where π_{av} is average firm profit. The denominator of the expression for D_π is positive because $M\pi_{av}$ is total profit, which is less than μ because the market is not fully covered (for $\mathcal{V}_0 > 0$).

Study of specific distributions

We now derive the distributions in Section 3: these involve parameter matching for the distribution examples.

Normal: For the normal, $F_X(x) = \frac{1}{\sigma\sqrt{2\tilde{\pi}}} \int_{-\infty}^{x} \exp\left(-\frac{(u-m)^2}{2\sigma^2}\right) du$, where $\tilde{\pi} = 3.1415...$. From Theorem 1, we have

$$F_\Pi(\pi) = F_X\left(\mu \ln\left(\frac{\pi D}{\mu}\right)\right),$$

where $\mu \ln\left(\frac{\pi D}{\mu}\right) \in (-\infty, \infty)$, so

$$F_\Pi(\pi) = F_X\left(\mu \ln\left(\frac{\pi D}{\mu}\right)\right) = \frac{1}{\sigma\sqrt{2\tilde{\pi}}} \int_{-\infty}^{\mu \ln\left(\frac{\pi D}{\mu}\right)} \exp\left(-\frac{(u-m)^2}{2\sigma^2}\right) du.$$

Using the change of variable $\Pi = \frac{\mu}{D}\exp\left(\frac{u}{\mu}\right)$ (so $u = \mu \ln\left(\frac{\Pi D}{\mu}\right)$ and $du = \frac{\mu}{\Pi}d\Pi$) we obtain

$$F_{\Pi}(\pi) = \frac{\mu}{\sigma\sqrt{2\tilde{\pi}}}\int_0^{\pi} \exp\left(-\frac{\left(\mu \ln\left(\frac{\Pi D}{\mu}\right) - m\right)^2}{2\sigma^2}\right)\frac{d\Pi}{\Pi},$$

which can be written in a standard form as:

$$F_{\Pi}(\pi) = \frac{1}{\left(\frac{\sigma}{\mu}\right)\sqrt{2\tilde{\pi}}}\int_0^{\pi} \frac{1}{\Pi}\exp\left(-\frac{\left(\ln \Pi - \left(\ln\left(\frac{D}{\mu}\right) - \frac{m}{\mu}\right)\right)^2}{2\left(\frac{\sigma}{\mu}\right)^2}\right)d\Pi.$$

Hence profits are log-normally distributed with parameters $\left[\ln\left(\frac{D}{\mu}\right) - \frac{m}{\mu}\right]$ and $\left(\frac{\sigma}{\mu}\right)$.

Recall: $D = M\int_{u \geq \underline{x}} \exp\left(\frac{u}{\mu}\right) f_X(u)\,du + \mathcal{V}_0$. Then:

$$D = \frac{M}{\sigma\sqrt{2\tilde{\pi}}}\int_{-\infty}^{\infty} \exp\left(\frac{x}{\mu}\right)\exp\left(\frac{-(x-m)^2}{2\sigma^2}\right)dx + \mathcal{V}_0.$$

Routine computation shows that:

$$D = M\exp\left(\frac{m}{\mu} + \frac{\sigma^2}{2\mu^2}\right) + \mathcal{V}_0.$$

Logistic.

A *logistic* distribution for quality-cost has a CDF given by $F_X(x) = \left(1 + \exp\left(-\frac{x-m}{s}\right)\right)^{-1}$, $x \in (\underline{x}, \infty)$, with mean m and variance $s^2\pi^2/3$. The PDF is similar in shape to the normal, but it has thicker tails (see the discussion in Fisk, 1961, and the comparison with the Weibull distribution). Hence, for $\mu > s$, profit $\Pi \in (0, \infty)$ is log-logistically distributed with parameters $\left(\frac{D}{\mu}\right)\exp\left(-\frac{m}{\mu}\right)$ and $\frac{\mu}{s}$:

$$F_{\Pi}(\pi) = \left(1 + \left(\frac{\pi}{\left(\frac{D}{\mu}\right)\exp\left(-\frac{m}{\mu}\right)}\right)^{-\frac{\mu}{s}}\right)^{-1}, \qquad \pi \in [0, \infty).$$

There is no closed form expression for D in this case. However, it can be shown that the condition $\mu > s$ guarantees that the output denominator D exists. The Log-logistic distribution (which provides a one parameter model for survival analysis) is very similar in shape to the log-normal distribution, but it has fatter tails. It has an explicit functional form, in contrast to the Log-normal distribution.

The logistic distribution (with mean m and standard deviation $s\tilde{\pi}/\sqrt{3}$) is given by:

$$F_X(x) = \frac{1}{1+\exp\left(-\frac{x-m}{s}\right)}, x \in (\underline{x}, \infty).$$

From Theorem 1,

$$F_\Pi(\pi) = F_X\left(\mu \ln\left(\frac{\pi D}{\mu}\right)\right) = \frac{1}{1+\exp\left(-\frac{\mu \ln\left(\frac{\pi D}{\mu}\right)-m}{s}\right)},$$

where $F_\Pi(0) = 0$ and $F_\Pi(\infty) = 1$. Thus

$$F_\Pi(\pi) = \frac{1}{1+\exp\left(\frac{m}{s}\right)\exp\left(\ln\left(\frac{\pi D}{\mu}\right)^{-\frac{\mu}{s}}\right)} = \frac{1}{1+\left(\frac{\pi}{\left(\frac{D}{\mu}\right)\exp\left(-\frac{m}{\mu}\right)}\right)^{-\frac{\mu}{s}}}.$$

Recall the log-logistic distribution is defined as:

$$F^{LL}(x;\alpha,\beta) = \frac{1}{1+\left(\frac{x}{\alpha}\right)^{-\beta}}, \quad x > 0.$$

Thus, the parameter matching is:

$$F_\Pi(\pi) = F^{LL}\left(x; \left(\frac{D}{\mu}\right)\exp\left(-\frac{m}{\mu}\right), \frac{\mu}{s}\right).$$

We need to check when D converges, i.e., when $\int_{-\infty}^{\infty} \exp\left(\frac{x}{\mu}\right) f_X(x)\,dx$ converges. Because

$$f_X(x) = \frac{1}{s}\frac{\exp\left(-\frac{x-m}{s}\right)}{\left(1+\exp\left(-\frac{x-m}{s}\right)\right)^2},$$

we need to ensure the convergence of the expression

$$\int_{-\infty}^{\infty} \frac{\exp\left(-x\left(\frac{1}{s}-\frac{1}{\mu}\right)\right)}{\left(1+\exp\left(-\frac{x-m}{s}\right)\right)^{2}} dx.$$

Convergence is guaranteed if and only if $\mu > s$.

Pareto: The Pareto distribution is given by: $F_X(x) = \frac{1-\left(\frac{\underline{x}}{x}\right)^{\alpha}}{1-\left(\frac{\overline{x}}{\underline{x}}\right)^{-\alpha}}$. From Theorem 1

$$F_{\Pi}(\pi) = \frac{1-\left(\frac{\underline{x}}{\mu\ln\left(\frac{\pi D}{\mu}\right)}\right)^{\alpha}}{1-\left(\frac{\overline{x}}{\underline{x}}\right)^{-\alpha}}.$$

Recall that $\pi = \frac{\mu\exp\left(\frac{x}{\mu}\right)}{D}$ or $x = \mu\ln\left(\frac{\pi D}{\mu}\right)$, so that $\underline{x} = \mu\ln\left(\frac{\underline{\pi} D}{\mu}\right)$ and $\overline{x} = \mu\ln\left(\frac{\overline{\pi} D}{\mu}\right)$, so that $F_{\Pi}(\underline{\pi}) = 0$ and $F_{\Pi}(\overline{\pi}) = 1$. D is bounded because the distribution of quality-cost is bounded.

Consider a log-Pareto distribution with scale parameter σ and shape parameters γ and β:

$$F^{LP}(\pi;\gamma,\beta,\sigma) = \frac{1-\left(1+\frac{1}{\beta}\ln\left(1+\frac{\pi-\underline{\pi}}{\sigma}\right)\right)^{-\frac{1}{\gamma}}}{1-\left(1+\frac{1}{\beta}\ln\left(1+\frac{\overline{\pi}-\underline{\pi}}{\sigma}\right)\right)^{-\frac{1}{\gamma}}}, \qquad \pi > \underline{\pi}.$$

In order to match parameters, of $F_{\Pi}(\pi)$ with $F^{LP}(\pi;\gamma,\beta,\sigma)$ observe that

$$\mu\ln\left(\frac{\pi D}{\mu}\right) = \mu\ln\left(\frac{\underline{\pi} D}{\mu}\frac{\pi}{\underline{\pi}}\right) = \underline{x} + \mu\ln\left(\frac{\pi}{\underline{\pi}}\right) = \underline{x} + \mu\ln\left(1+\frac{\pi-\underline{\pi}}{\underline{\pi}}\right).$$

Therefore:

$$F_{\Pi}(\pi) = \frac{1-\underline{x}^{\alpha}\left(\mu\ln\left(\frac{\pi D}{\mu}\right)\right)^{-\alpha}}{1-\left(\frac{\overline{x}}{\underline{x}}\right)^{-\alpha}} = \frac{1-\left(1+\frac{\mu}{\underline{x}}\ln\left(1+\frac{\pi-\underline{\pi}}{\underline{\pi}}\right)\right)^{-\alpha}}{1-\left(\frac{\overline{x}}{\underline{x}}\right)^{-\alpha}}.$$

Thus π obeys a Log-Pareto distribution $F^{LP}\left(\pi;\frac{1}{\alpha},\frac{\underline{x}}{\mu},\underline{\pi}\right)$, i.e., $\gamma = \frac{1}{\alpha}, \beta = \frac{\underline{x}}{\mu}, \sigma = \underline{\pi}$.

It remains to check that the normalization factors are equal. Recall that $\overline{\pi}/\underline{\pi} = \exp\frac{\overline{x}-\underline{x}}{\mu}$.

Using the specification $\gamma = \frac{1}{\alpha}, \beta = \frac{\underline{x}}{\mu}, \sigma = \underline{\pi}$, we get:

$$\begin{aligned}\left(1 + \frac{\mu}{\underline{x}} \ln\left(1 + \frac{\overline{\pi} - \underline{\pi}}{\underline{\pi}}\right)\right)^{-\alpha} &= \left(1 + \frac{\mu}{\underline{x}} \ln\left(\frac{\overline{\pi}}{\underline{\pi}}\right)\right)^{-\alpha} \\ &= \left(1 + \left(\frac{\overline{x} - \underline{x}}{\overline{x}}\right)\right)^{-\alpha} = \left(\frac{\overline{x}}{\underline{x}}\right)^{-\alpha}.\end{aligned}$$

Figure 1: Increasing cost to quality-cost relation.

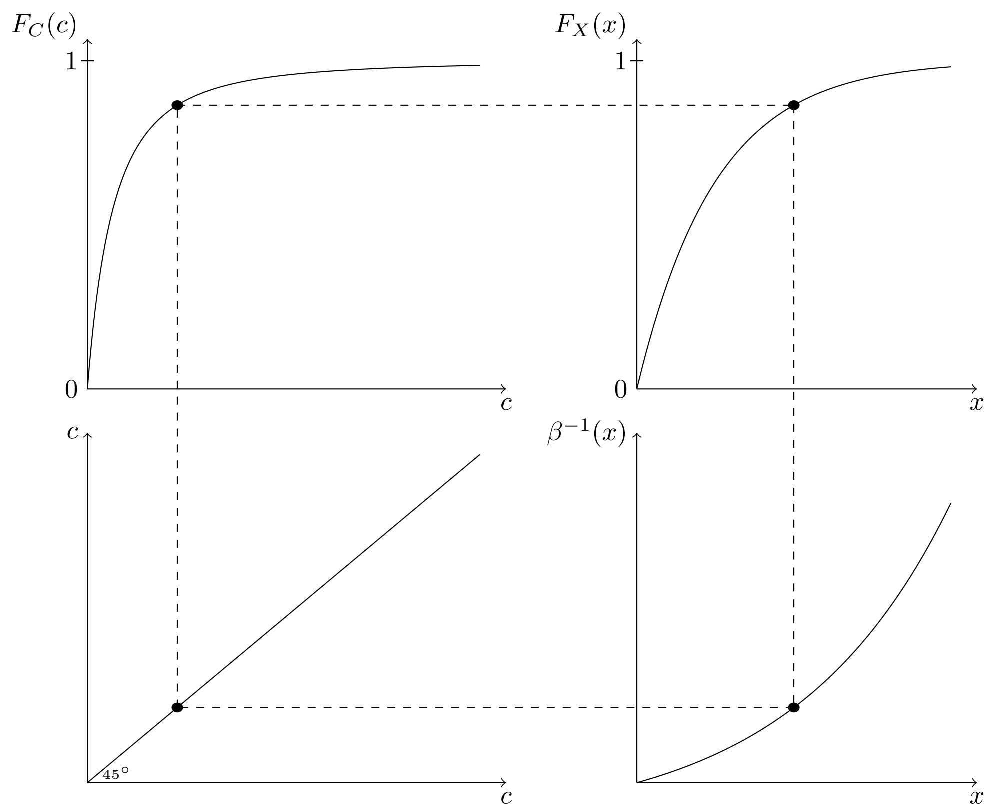

Figure 2: Hump relation between cost and quality-cost.